Spectrum and Continuum Eigenfunctions of Schrödinger Operators

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We consider Schrödinger operators $H = -\frac{1}{2}A + V$ for a large class of potentials $V$. We show that if $H\varphi = E\varphi$ has a polynomially bounded solution $\varphi$ then $E$ is in the spectrum of $H$. This is accomplished by proving that the spectrum of $H$ as an operator on $L^2$ is identical to its spectrum as an operator on the weighted $L^2$ space.

1. INTRODUCTION

In this paper, we want to discuss eigenfunctions of Schrödinger operators

$$H = -\frac{1}{2}A + V$$

on $L^2(R^n)$. We will deal with a wide class of potentials; typically we will require that $V = V_+ - V_-$ with $V_+ \geq 0$ and $V_+ \in K_{loc}^\infty$, $V_- \in K^\infty$, where $K^\infty$ is the class discussed in [2]:

**DEFINITION.** If $v \geq 3$, we say $f \in K_v$ if and only if

$$\lim_{\alpha \to 0} \sup_x \int_{|x-y| \leq \alpha} |x - y|^{-(v-2)} |f(y)| d^v y = 0.$$  

If $v = 2$, we replace $|x - y|^{-(v-2)}$ by $\ln |x - y|^{-1}$ (and take $\alpha \leq 1$). If $v = 1$, $K_v$ is the set of $f$'s with

$$\sup_x \int_{|x-y| \leq 1} |f(y)| dy < \infty,$$

$f \in K_v^{loc}$ if and only if $fg \in K_v$ for all $g \in C_0^\infty(R^n)$.

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This class is sufficiently large to include virtually all interesting $V$'s which lead to $H$'s which are bounded below. It is convenient since a Harnack inequality holds for such potentials [2].

We are interested in solutions, $\varphi$, of

$$H\varphi = E\varphi$$

(1.2)

(we will discuss the sense in which this holds below) and, in particular, for which $E$ (1.2) has a polynomially bounded solution.

One direction is already well known. Generalized eigenfunction expansions of differential operators is a subject associated with Berezanski, Gårding, Gel'fand, Kac and Maurin (see Berezanski [3] for references; the implementation of their ideas for Schrödinger operators is described in Faris [5], Herbst-Sloan [6] and Kovalenko-Semonov [7] and reviewed in Simon [12]. One consequence of this theory is the following (see [7] or [12]).

**Theorem 1.1.** Let $V_+ \in K_{\text{loc}}^1$, $V_- \in K_1$ and let $H$ be the associated $L^2$-Schrödinger operator. Then for every $\varepsilon > 0$, (1.2) has a distributional solution $\varphi$ obeying

$$|\varphi(x)| \leq C(1 + |x|)^{(1/2)N + \varepsilon}$$

(1.3)

for $H$-spectrally almost every $E$. In particular, the set of $E$ for which (1.2) has a solution obeying (1.3) is dense in the spectrum of $H$.

**Remarks.**

1. If $V$ obeys the above hypothesis, then $H$ defines a closed quadratic form on $Q(-\Delta) \cap Q(V_+)$ (and $C_0^\infty(\mathbb{R}^n)$ is a form core [10]) and the associated self-adjoint operator is what we mean by “the associated $L^2$-Schrödinger operator.”

2. If $A$ is a self-adjoint operator, we say something holds “$A$-spectrally almost everywhere” if the set $A$ for which it holds has an associated spectral projection which is the identity.

Our goal in this paper is to consider the converse of this result, i.e., to show that if (1.2) has a solution obeying

$$|\varphi(x)| \leq C(1 + |x|)^N$$

(1.4)

for some $N$, then $E$ is in the spectrum of $H$. Surprisingly, except for [13, 14], this appears not to have been discussed before. We note that if polynomial growth is replaced by exponential growth, the result is not true as consideration of the case $V = 0$ shows. We also note that if $\varphi$ obeys (1.2) and

$$\int_{|x-y| < 1} |\varphi(y)| \, d^n y \leq C(1 + |x|)^N$$

(1.5)

then automatically (1.4) holds by a Harnak type inequality; see [2].
In Section 3, we will prove

**Theorem 1.2.** Let \( V_+ \in K_{p}^{\text{loc}} \), \( V_- \in K_p \) and \( V \in L_{1}^{2} \) and let \( H \) be the associated \( L^{2} \)-Schrödinger Operator. Suppose that (1.2) has a distributional solution, \( \varphi \), obeying (1.4) for some \( N \). Then \( E \in \text{spec}(H) \).

This, together with Theorem 1.1 immediately implies

**Theorem 1.3.** If \( V \) obeys the hypothesis of Theorem 1.2, then

\[
\text{spec } H - \{ E | (1.2) \text{ has a distributional solution obeying (1.4)} \}.
\]

Actually, the condition \( V \in L_{1}^{2} \) is only needed for a "nice" meaning to the expression "solution of (1.2)." If \( V_+ \in K_{p}^{\text{loc}} \), \( V_- \in K_p \), then (see Section 2) \( e^{-itH} \) defines a map with

\[
\|(e^{-itH}g)(x)\| \leq Ce^{-ax^2}
\]

for every \( g \in C_{0}^{\infty} \), so that \( (\varphi, e^{-itH}g) \) makes sense if \( \varphi \) obeys (1.4). In Section 2, we will prove

**Theorem 1.4.** Let \( \varphi \) obey (1.4). Let \( V_+ \in K_{p}^{\text{loc}} \), \( V_- \in K_p \), and suppose that \( \varphi \) "obeys (1.2)" in the sense that

\[
(\varphi, e^{-itH}g) = e^{-itE}(\varphi, g)
\]

for every \( g \in C_{0}^{\infty} \). Then \( E \in \text{spec}(H) \).

The basic methods we use in Section 2 involve another natural question which has not been previously considered. Let \( L_{2}^{\delta} = \{ f \mid (1 + x^2)^{\delta/2} f \in L^{2} \} \) with the norm

\[
\| f \|_{\delta} = \left( \int (1 + x^2)^{\delta} |f(x)|^2 \, dx \right)^{1/2}
\]

In Section 2, we will prove that for \( g \in C_{0}^{\infty} \):

\[
\| e^{-itH}g \|_{\delta} \leq Ce^{\delta} \| g \|_{\delta}
\]

so that the semigroup \( e^{-itH} \) can be defined on \( L_{2}^{\delta} \). We denote its generator by \( H_{\delta} \). In Section 2, we will prove

**Theorem 1.5.** Let \( V_- \in K_p \), \( V_+ \in K_{p}^{\text{loc}} \). Then, for any \( \delta \)

\[
\text{spec } (H_{\delta}) = \text{spec } (H).
\]

For \( V \)'s going to zero at infinity sufficiently rapidly \( H_{\delta} \) have been exten-
sively considered in the Agmon [1]–Kuroda [8] theory. For this case, results
close to this appear, for example, in Reed–Simon [9].

The point is that (1.7) implies that $H_\delta \varphi = E \varphi$ in operator for suitable $\delta$.
Thus $E$ is in the point spectrum of $H_\delta$ and so by (1.9) in $\text{spec}(H)$. Thus
Theorem 1.5 implies Theorem 1.4.

In Section 3, we prove Theorem 1.2 from Theorem 1.4 by proving $C_0^\omega$ is
an operator core for $H_\delta$. In Section 4 we discuss $H$ on the weighted space
$L^2(\mathbb{R}^n, e^{-a|x|} \, dx)$, where the spectrum changes but, in particular, we prove
that (1.4) for some $N$ can be replaced by

$$|\varphi(x)| \leq C_0 e^{a|x|}$$

for all $a > 0$.

Some special cases of Theorems 1.1 and 1.3 can be found in [13, 14].
Babbitt calls our Theorem 1.3 “a tight rigging.”

It is a pleasure to thank B. Souillard for raising the question of converses
to Theorem 1.1.

2. SPECTRUM ON POLYNOMIALLY WEIGHTED SPACES

The basic facts that we will use about Schrödinger semigroups are:

(i) $\exp(-tH)$ has an integral kernel bounded by

$$C_t e^{-(\nu + \varepsilon)t} \exp(\frac{A}{t})$$

for all $\varepsilon$, some $A$ and $C$ depending on $\varepsilon$.

(ii) If $H_\delta = -\frac{1}{2}A$ and $H' = H_\delta + 2V$ and $(e^{-tH})(x, y)$ is the integral
kernel for $e^{-tH}$, then

$$e^{-tH}(x, y) \leq \left[ e^{-tH_\delta}(x, y) \right]^{1/2} \left[ e^{-tH'}(x, y) \right]^{1/2}.$$  \hfill (2.2)

Claim (i) follows from the Dunford–Pettis theorem and the fact that the
semigroup is bounded from $L^1$ to $L^\infty$ with norm given by (2.1), [4, 11]. Fact
(ii) is just the Schwartz inequality in path space [11].

From (2.2), we conclude that for $f \in C_0^\infty, f \geq 0$:

$$(1 + x^2)^\delta e^{-tH}[(1 + x^2)^{-\delta} f]$$

$$\leq (e^{-tH'} f)^{1/2} \left( (1 + x^2)^{2\delta} e^{-tH_\delta} (1 + x^2)^{-2\delta} f \right)^{1/2}$$

pointwise, so that if $\|A\|_{\delta, \delta}$ is the norm if $A$ is a map of $L_2^\delta$ to itself, then

$$\|e^{-tH}\|_{\delta, \delta} \leq \|e^{-tH'}\|_{\delta, \delta}^{1/2} \|e^{-tH_\delta}\|_{\delta, \delta}^{1/2}$$ \hfill (2.3)
(this can also be proven by complex interpolation). Using the explicit integral kernel of \( e^{-\delta t_0} \) and the fact that
\[
\| f * g \|_2 \leq \| f \|_2 \| g \|_1
\]
(with \(*\) convolution), one easily sees that for \( \delta > 0 \):
\[
\| e^{-\delta t_0} \|_{2,\infty} \leq 1 + Ct^{2\delta}
\]
so that we have

**Proposition 2.1.** If \( V_+ \in K_{t}^{loc}, V_- \in K_t \), then \( e^{-\delta t} \) defines a bounded map of \( L^2_{\delta} \) to itself obeying
\[
\| e^{-\delta t} \|_{\delta,\infty} \leq Ce^{\delta}.
\]
(2.4)

In many ways, the main result of this paper is

**Theorem 2.2 (\textit{\textit{\textit{\textit{Theorem 1.5}}}}).** The generator, \( H_\delta \), of \( e^{-\delta t} \) on \( L^2_{\delta} \) obeys
\[
\text{spec} (H_\delta) = \text{spec} (H)
\]

**Proof.** As a preliminary, we note that \( Q(H) \subset Q(H_0) \) so
\[
(H - z)^{-1} \tilde{\mathcal{V}} \quad (2.5)
\]
is bounded as an operator on \( L^2 \) for any \( z \notin \text{spec} (H) \).

Now, let \( \tilde{\cdot} \) denote the normal conjugation and note that, with \( A' = (A^*) \) we have
\[
H = H^* = \tilde{H} = H'
\]
so that
\[
H'_\delta = H_{-\delta}
\]
and thus
\[
\{(H_\delta - z)^{-1}\}' = (H_{-\delta} - z)^{-1}
\]
so by interpolation, we see that if \( z \notin \text{spec} (H_\delta) \), then \( z \notin \text{spec} (H) \).

To prove the result, we only need to show that if \( z \notin \text{spec}(H) \), then \( z \notin \text{spec} (H_\delta) \). By the interpolation and duality argument, we can suppose that \( \delta \) is positive integer. We give formal commutator manipulations which
are easy to justify. Let $g = (1 + x^2)^{\delta/2}$ and we proceed inductively in $\delta$ (which is assumed integral). Then, if $z \notin \text{spec } (H)$:

$$g(H - z)^{-1} g^{-1} = (H - z)^{-1} + \lfloor g, (H - z)^{-1} \rfloor g^{-1}$$

$$= (H - z)^{-1} - \frac{1}{2} (H - z)^{-1} \Delta g(H - z)^{-1} g^{-1} \quad (2.6)$$

$$+ (H - z)^{-1} \nabla \cdot (\nabla g)(H - z)^{-1} g^{-1}$$

By induction in $\delta$, $\Delta g(H - z)^{-1} g^{-1}$ and $\nabla g(H - z)^{-1} g^{-1}$ are bounded on $L^2$, so by (2.5), $g(H - z)^{-1} g^{-1}$ is bounded on $L^2$, i.e., $(H - z)^{-1}$ is bounded on $L^2_\delta$.

As noted before, this implies:

**THEOREM 2.3 (≡ Theorem 1.4).** Let $V_+ \in \mathcal{K}_p^{\text{loc}}$, $V_- \in \mathcal{K}_-$ and suppose $\phi \in L^2_{-N}$ for some $N$ obeys

$$H_{-N} \phi = E \phi$$

then $E \in \text{spec } (H)$.

In the next section, we will need a small extension of the above argument:

**THEOREM 2.4.** Let $\delta > 0$. Suppose that $\phi \in L^2_{\delta}$ lies in $D(H_\delta)$. Then $\nabla \phi \in L^2_{\delta}$.

**Proof:** Let $g = (1 + x^2)^{\delta/2}$. For simplicity suppose $-1 \notin \text{spec } (H)$. Writing

$$g \nabla \phi = g \nabla (H + 1)^{-1} g^{-1} \lfloor g(H + 1) \phi \rfloor$$

and noting that $g(H + 1) \phi \in L^2$ by hypothesis we see that it suffices that $g \nabla (H + 1)^{-1} g^{-1}$ is bounded on $L^2$. Since

$$\lfloor g, \nabla \rfloor (H + 1)^{-1} g^{-1} = -(\nabla g)(H + 1)^{-1} g^{-1}$$

is bounded by Theorem 2.2, we see that it suffices that $\nabla g(H + 1)^{-1} g^{-1}$ is bounded. But, by (2.6):

$$\nabla g(H + 1)^{-1} g^{-1} = \nabla (H + 1)^{-1} - \frac{1}{2} \nabla (H + 1)^{-1} (\Delta g)(H + 1)^{-1} g^{-1}$$

$$+ \nabla (H + 1)^{-1} \nabla \cdot (\nabla g)(H + 1)^{-1} g^{-1}$$

and each term is bounded on $L^2$ since $\nabla (H + 1)^{-1}$, $\nabla (H + 1)^{-1} \nabla$ and $g(H + 1)^{-1} g^{-1}$ are all bounded. ■
3. Cores on Polynomially Weighted Spaces

To relate distributional solutions of (1.2) to operator solutions we will need the following result which follows the "semigroup version of Kato's inequality" [10]; we remark that because of the \( V_{-} \) possibility, this proof is new even in the case \( \delta = 0 \):

**Theorem 3.1.** Let \( V_{+} \in K_{p}^{\text{loc}}, \ V_{-} \in K_{r}, \ V \in L_{\text{loc}}^{1} \) and let \( \delta \geq 0 \). Then \( C_{0}^{\infty} \) is a core for \( H_{\delta} \).

**Proof.** \( e^{-iH_{\delta}t} \) is bounded and \( e^{-izH} \) (for \( \delta = 0 \)) is holomorphic, so by the Stein interpolation theorem, \( e^{-iH_{\delta}t} \) is a holomorphic semigroup so \( \text{Ran} \ (e^{-iH_{\delta}}) \subset D(H_{\delta}) \) and if \( \varphi \in D(H_{\delta}) \), then \( H_{\delta}e^{-iH_{\delta}t}\varphi \rightarrow H_{\delta}\varphi \) in \( L_{\delta}^{2} \). Since \( \text{Ran} \ (e^{-iH_{\delta}}) \subset L_{\infty}^{\infty} \), we conclude that \( L_{\infty}^{\infty} \cap D(H_{\delta}) \) is a core for \( H_{\delta} \). Let \( g \in C_{0}^{\infty} \) with \( 0 \leq g \leq 1 \) and \( g \equiv 1 \) near \( x = 0 \). Let \( g_{n}(x) = g(x/n) \). Let \( \varphi \in L_{\infty}^{\infty} \cap D(H_{\delta}) \). Then

\[
g_{n}\varphi \rightarrow \varphi
\]

in \( L_{\delta}^{2} \) and

\[
H(g_{n}\varphi) = g_{n}H\varphi + (\Delta g_{n})\varphi + 2(\nabla g_{n}) \cdot \nabla g \rightarrow H\varphi
\]

in \( L_{\delta}^{2} \) since \( \nabla \varphi \in L_{\delta}^{2} \) by Theorem 2.4.

As a result, \( L_{\infty}^{\infty} \cap D(H_{\delta}) \cap (\text{compact support}) \) is a core for \( H_{\delta} \). Now mollify and use \( V \in L_{\text{loc}}^{1}, \varphi \in L_{\infty}^{\infty} \) to approximate by functions in \( C_{0}^{\infty} \).

**Theorem 3.2** (\( \equiv \) Theorem 1.2). Let \( V_{+} \in K_{p}^{\text{loc}}, \ V_{-} \in K_{r}, \ V \in L_{\text{loc}}^{1} \). Let \( \varphi \in L_{\text{loc}}^{1} \) for some \( \delta > 0 \) and obey

\[
(\varphi, Hg) = E(\varphi, g)
\]

for all \( g \in C_{0}^{\infty} \) (i.e., \( H\varphi = E\varphi \) in distributional sense). Then \( \varphi \in D(H_{\delta}) \).

**Proof.** Since \( C_{0}^{\infty} \) is a core for \( H_{\delta} \), (3.1) holds for all \( g \in D(H_{\delta}) \). Thus \( \varphi \in D(H_{\delta}^{*}) \) but \( H_{\delta}^{*} = H_{\delta} \).

4. Spectrum on Exponentially Weighted Spaces

In this final section, we want to prove the following which generalizes Theorem 2.2:

**Theorem 4.1.** Let \( V_{+} \in K_{p}^{\text{loc}}, \ V_{-} \in K_{r} \). Then there exists a \( D \)
depending only on $V$ so that if (1.2) has a solution (in the sense $(\varphi, e^{-tH}g) = e^{-t\varphi}(\varphi, g)$ for all $g \in C_0^\infty$) obeying
\[
\int e^{-2a|x|} |\varphi(x)|^2 \, dx < \infty \tag{4.1}
\]
then
\[
\text{dist}(E, \text{spec}(H)) \leq 2Da(|E|^{1/2} + 1) + |a^2 + va| \tag{4.2}
\]
In particular, if (4.1) holds for all $a > 0$, then $E \in \text{spec}(H)$.

**Proof.** As in Section 2, $e^{-tH}$ defines an exponentially bounded semigroup on $L^2(\omega) = \{f : \int |f|^2 e^{-2a|x|} \, dx < \infty \}$. We need only show that if $E$ fails to obey (4.2), and $E \notin \text{spec}(H)$, then $g(H - E)^{-1}$ is bounded on $L^2$, where
\[
g(x) = \exp(-a \sqrt{x^2 + 1})
\]
Let $\varphi \in C_0^\infty$, let $\alpha = \|g(H - z)^{-1}g^{-1}\varphi\|$ (which is finite) and apply (2.6) to $\varphi$ to get
\[
\alpha \leq \|(H - z)^{-1}\| \|\varphi\| + Q\alpha
\]
with
\[
Q = \|(H - z)^{-1}\| \|\Delta g\| g^{-1} \|_{\infty} + 2 \|(H - z)^{-1}\| \nabla \| \|\nabla g\| g^{-1} \|_{\infty}.
\]
But by elementary calculations
\[
\|\Delta g\| g^{-1} \|_{\infty} \leq a^2 + va,
\]
\[
\|\nabla g\| g^{-1} \|_{\infty} \leq a
\]
and
\[
\|(H - z)^{-1}\| \nabla \| \leq D(\|z\| + 1)^{1/2} \| \text{dist}(z, \text{spec}(H))\|^{-1},
\]
where $D$ is $H$-dependent. If $Q < 1$, we see that $g(H - z)^{-1}g$ is bounded. \hfill \qed

*Notes added in proof.*

1. J. Rauch has pointed out that we neglected to prove that $(H - z)^{-1}$ is the inverse to $H_{\delta} - z$. This can be proven as follows: by our semigroup definition of $H_{\delta}$, it is true if $\text{Re}z$ is very negative and thus by analyticity, and by the fact that the $L^2$ spectrum of $H$ is a subset of $R$, the result is true for all $z$ in the resolvent set of $H$.

2. A slightly stronger result than Theorems 1.2, 1.3 holds; namely if the eigenfunction is not in $L^2$, then $E$ must lie in the essential spectrum of $H$. For our argument shows that if $E$ is an isolated point of $\text{spec}(H)$, the projection $P = \int_{|z - E| = \epsilon} (H - z)^{-1} \, dz/(2\pi i)$ is bounded
from $L^2_a$ to $L^2_b$ and is the spectral projection for any $H$. If its range is finite dimensional, then $P[L^2_a] = P[L^2_b]$ so any function in $P[L^2_a]$ is in $L^2$.

1. M. Glazman's book "Direct Methods of the Qualitative Spectral Analysis of Differential Operators," Davey & Co., 1965. By using results from [2], one can extend Schrödinger's method to treat the general case discussed in this paper and the extension noted in Remark 2. above. Schrödinger's proof is more direct than ours.

REFERENCES