MATRIX A_p WEIGHTS VIA MAXIMAL FUNCTIONS

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Abstract. The matrix A_p condition extends several results in weighted norm theory to functions taking values in a finite-dimensional vector space. Here we show that the matrix A_p condition leads to L^p-boundedness of a Hardy-Littlewood maximal function, then use this estimate to establish a bound for the weighted L^p norm of singular integral operators.

1. Preliminaries

Weighted Norm theory forms a basic component of the study of singular integrals. Here one attempts to characterize those measure spaces over which a broad class of singular integral operators remain bounded. For the case of singular integral operators on $\mathbf{C}$-valued functions in Euclidean space, the answer is given by the Hunt-Muckenhoupt-Wheeden theorem [10]. It states that the necessary and sufficient condition for boundedness in $L^p(d\mu)$ is that $d\mu = W(x)\,dx$ and the function W satisfies the A_p condition, namely: $\Big(\frac{1}{|B|}\int_B W\,dx\Big)^{1/p}\Big(\frac{1}{|B|}\int_B W^{-p'/p}dx\Big)^{1/p'} \leq C$ for all balls $B \subset \mathbf{R}^n$.

The A_p condition requires considerable interpretation in order to apply it to weighted measures of $\mathbf{C}^d$-valued functions. First, the weight $W(x)$ should take values in the space of positive $d\times d$ Hermitian forms. This raises concerns about the order in which products are taken, since matrices need not commute, and also what it means for the quantity on the left-hand side to be uniformly bounded. Treil [21] conjectured that the correct statement of the matrix A_2 condition should be

$$\sup_B \Big\| \Big(\frac{1}{|B|}\int_B W\,dx\Big)^{1/2}\Big(\frac{1}{|B|}\int_B W^{-1}dx\Big)^{1/2}\Big\| < \infty$$

where exponents $1/2$ indicate operator powers of a nonnegative matrix. This was subsequently proven in [23] and again in [24].

If p is different from 2, the matrix A_p condition cannot be written in terms of averages of operator powers of weight W. Averages still play a crucial role, however it is more accurate to regard $W(x)$ as a Banach space norm on $\mathbf{C}^d$ rather than a matrix. A correct formulation of the matrix A_p condition, which is also the subject of this note, first appeared in [12] and [25]. Because their statements do not appear similar, it is especially important to understand what properties matrix A_p weights share with their scalar counterparts. This is discussed further in the next section.

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Boundedness estimates on singular integral operators were originally obtained by way of the Hardy-Littlewood maximal function M. If a scalar weight W possesses the A_∞ property (several equivalent definitions are given in [18]), then the L^p norm of any singular integral is dominated by the L^p norm of M via a distributional argument commonly known as the good-λ inequality. The A_p condition is specifically required to ensure that $\|Mf\|_{L^p(W)} \leq C\|f\|_{L^p(W)}$.

Some of these techniques fail to generalize to the case of vector-valued functions with matrix weights. There is no known analogue of the A_∞ property to create simultaneous estimates for every exponent p. The weak-$L^p(W)$ spaces used to prove boundedness of the Hardy-Littlewood maximal function are not well defined in this setting. In general, much of the ability to compare objects and dominate one by another is lost when the objects are vectors rather than scalars. The theory of matrix weights has consequently evolved along much different lines. One fundamental technique employed in both [23] and [25] is to choose a good basis (often inspired by Haar functions) in $L^p(W)$ and consider the integral operator as a matrix acting on the coefficient space. Estimates may then be made separately on the matrix and on the coefficient embedding operator. Even in the scalar case these ideas have yielded new results and new ways of approaching weighted norm problems.

In this note we attempt to tackle the difficulties of extending the classical theory, or else circumvent them. Some arguments may be borrowed nearly word for word, some remain intact only if they are presented in a specific manner. Our hope is to discover which properties of scalar A_p weights admit some generalization to the case of vector-valued functions and matrix weights, leading to a more complete understanding of the matrix A_p class.

Let T be a singular integral operator associated to kernel $K(x)$ in the sense that $Tf(x) = \int_{\mathbf{R}^n} K(x-y)f(y)\,dy$ for almost every x outside the support of f. The following regularity hypotheses are to be assumed for K:

$$|K(x)| \leq C|x|^{-n} \text{ and } |\nabla K(x)| \leq C|x|^{-n-1} \tag{1}$$

and additionally we suppose that for some choice of p, $1 < p < \infty$, the bound $\|Tf\|_{L^p} \leq A\|f\|_{L^p}$ holds for all $f \in L^p$. One may then apply T to functions taking values in $\mathbf{C}^d$ by allowing it to act separately on each coordinate function, that is: $(Tf)_j = Tf_j$. This new operator, also denoted by T, is a singular integral operator whose associated convolution kernel is K times the identity matrix.

In a similar manner, define the truncated operators T_ϵ to be convolution with $K_\epsilon(x) = \chi_{\{|x|>\epsilon\}}K(x)$ for all $\epsilon > 0$. Note that T and the T_ϵ all commute with pointwise multiplication by any constant matrix Λ, in other words $\Lambda Tf = T(\Lambda f)$.

A matrix weight W is a function on $\mathbf{R}^n$ taking values in $d \times d$ positive-definite matrices, with weighted norm space $L^p(W)$ defined by

$$\|f\|^p_{L^p(W)} = \int_{\mathbf{R}^n} |W^{1/p}f|^p dx \tag{2}$$

One is often concerned with the relationship between a weight and its average over arbitrary balls. The most straightforward notion of an average, $W_B = \frac{1}{|B|}\int_B W\,dx$, turns out to be useful only in the study of $L^2(W)$. With any exponent $p \neq 2$, this does not properly respect the structure of the underlying L^p-space. The following definitions are needed instead:

A *metric* $\rho = \rho_x(\,\cdot\,)$ denotes a family of Banach space norms on $\mathbf{C}^d$, indexed by $x \in \mathbf{R}^n$. The weighted norm space $L^p(\rho)$ is given by

$$\|f\|^p_{L^p(\rho)} = \int_{\mathbf{R}^n} \big[\rho_x(f(x))\big]^p \, dx$$

Note that for any matrix weight W, $L^p(W)$ is isometrically equivalent to $L^p(\rho)$ with the metric $\rho_x(\mathbf{e}) = |W^{1/p}(x)\mathbf{e}|$. Given a ball $B \subset \mathbf{R}^n$ and an exponent $p > 1$, let $\rho_{p,B}$ be defined by the formula

$$\rho_{p,B}(\mathbf{e}) = \Big(\frac{1}{|B|}\int_B \big[\rho_x(\mathbf{e})\big]^p \, dx\Big)^{1/p}$$

This will be our method for averaging the metric ρ over a ball B.

The dual metric ρ^* is defined pointwise in x to be

$$\rho^*_x(\mathbf{e}) = \sup_{\mathbf{f} \in \mathbf{C}^d} \frac{|(\mathbf{e}, \mathbf{f})|}{\rho_x(\mathbf{f})}$$

One immediate consequence is that $(\mathbf{e}, \mathbf{f}) \leq \rho^*_x(\mathbf{e})\rho_x(\mathbf{f})$.

Proposition 1.1. *For any $\mathbf{e} \in \mathbf{C}^d$ and any ball $B \subset \mathbf{R}^n$, $\rho^*_{p',B}(\mathbf{e}) \geq (\rho_{p,B})^*(\mathbf{e})$.*

Proof. Given two vectors $\mathbf{e}, \mathbf{f} \in \mathbf{C}^d$,

$$\begin{aligned}(\mathbf{e}, \mathbf{f}) &\leq \frac{1}{|B|}\int_B \rho^*_x(\mathbf{e})\rho_x(\mathbf{f})\, dx \\ &\leq \Big(\frac{1}{|B|}\int_B \big[\rho^*_x(\mathbf{e})\big]^{p'} dx\Big)^{1/p'} \cdot \Big(\frac{1}{|B|}\int_B \big[\rho_x(\mathbf{f})\big]^{p} dx\Big)^{1/p} \\ &= \rho^*_{p',B}(\mathbf{e})\rho_{p,B}(\mathbf{f})\end{aligned}$$

In other words, $\rho^*_{p',B}(\mathbf{e}) \geq \dfrac{(\mathbf{e}, \mathbf{f})}{\rho_{p,B}(\mathbf{f})}$. The proof is completed by taking the supremum over all $\mathbf{f} \in \mathbf{C}^d$. □

A metric ρ is called an A_p *metric* if there exists some constant $C < \infty$ so that the opposite statement

$$\rho^*_{p',B}(\mathbf{e}) \leq C(\rho_{p,B})^*(\mathbf{e}) \quad \text{for all balls } B \subset \mathbf{R}^n \tag{3}$$

is also true. Since the averages over cubes and balls in $\mathbf{R}^n$ differ by no more that a fixed constant, A_p metrics satisfy an analogous condition for cubes, and vice versa. Stated either way, the A_p condition characterizes an important class of weighted measures.

Theorem 1. (Nazarov,Treil [12], Volberg [25]) *Let $d < \infty$. The following statements are equivalent:*

1) *The Hilbert Transform is bounded on $L^p(\rho)$.*
2) *ρ is an A_p metric.*

We will prove this theorem again for metrics which are induced by some matrix weight W. There is no loss of generality because for fixed dimension $d < \infty$ every metric can be uniformly approximated by matrix weights.

Proposition 1.2. *Let $d < \infty$. Given a Banach space norm ρ_x on $\mathbf{C}^d$, there exists a positive selfadjoint matrix W_x such that*

$$\rho_x(\mathbf{e}) \le |W_x(\mathbf{e})| \le \sqrt{d} \cdot \rho_x(\mathbf{e}) \quad \text{for all } \mathbf{e} \in \mathbf{C}^d. \tag{4}$$

Proof. Let O represent the unit ball of ρ_x, and E the ellipsoid of maximal volume contained in O. There exists a positive selfadjoint matrix W_x such that $W_x(E)$ is the standard unit ball in $\mathbf{C}^d$. The image $W_x(O)$ is a convex balanced set containing the unit ball, and containing no ellipsoid of greater volume.

If there exists a point $\mathbf{v} \in W_x(O)$ with $|\mathbf{v}| > \sqrt{d}$, then by convexity the boundary of $W_x(O)$ can only be tangent to the unit sphere at points $\mathbf{w}$ such that

$$(\mathbf{w}, \mathbf{v}) \le \frac{1}{|\mathbf{v}|} < \frac{1}{\sqrt{d}}$$

For some $\delta > 0$ the ellipsoid with major axis length e^{δ} in the direction of $\mathbf{v}$ and minor axes length $e^{-\delta/(|\mathbf{v}|^2-1)}$ in every direction perpendicular to $\mathbf{v}$ is also contained in $W_x(O)$. This has strictly greater volume than the unit ball, contradicting the property of $W_x(O)$ stated above. □

It is now possible to state the A_p condition in terms of matrix weights, though some precision is lost in the process. Given a matrix weight W and a ball $B \subset \mathbf{R}^n$, define a Banach space norm X_B on $\mathbf{C}^d$ by considering the $L^p(W)$ norm of characteristic functions on B.

$$\|\mathbf{v}\|_{X_B} = |B|^{-1/p} \|\chi_B \mathbf{v}\|_{L^p(W)}$$

By proposition 1.2 there exists a positive-definite $d \times d$ matrix V_B such that $\|\mathbf{v}\|_{X_B} \le |V_B \mathbf{v}| \le d^{1/2} \|\mathbf{v}\|_{X_B}$. From a heuristic standpoint, V_B might be considered an "L^p average" of $W^{1/p}$ over ball B. With $p' = \frac{p}{p-1}$ the dual exponent to p, let V'_B be an $L^{p'}$ average of $W^{-1/p}$. In summary, matrices V_B, V'_B enjoy the following properties:

$$\begin{aligned} |V_B \mathbf{v}| &\sim |B|^{-1/p} \|\chi_B W^{1/p} \mathbf{v}\|_{L^p} \\ |V'_B \mathbf{v}| &\sim |B|^{-1/p'} \|\chi_B W^{-1/p} \mathbf{v}\|_{L^{p'}} \end{aligned} \tag{5}$$

Remark. The definition of V_B and V'_B depends implicitly on the method used to approximate Banach space norms by matrices. For the purposes of our discussion, V_B and V'_B may be any two matrices satisfying (5).

The statement about weights taking the place of proposition 1.1 is

$$|V_B V'_B \mathbf{e}| \ge |\mathbf{e}| \text{ for all vectors } \mathbf{e} \in \mathbf{C}^d \text{ and balls } B \subset \mathbf{R}^n.$$

A matrix weight W satisfies the *matrix A_p condition* if $V_B V'_B$ are uniformly bounded as operators on $\mathbf{C}^d$; that is

$$\|V_B V'_B\| \le C < \infty \quad \text{for all balls } B \subset \mathbf{R}^n \tag{6}$$

The exact value of C depends on the choice of V_B and V'_B, and is therefore determined here only up to a factor of d.

Our approach to Theorem 1 is styled after Coifman and Fefferman's proof [5] in the scalar ($d = 1$) case. Two technical problems arise immediately: first that general $d \times d$ matrices do not commute with one another, and second the matter of defining a maximal operator for vector-valued functions. To choose pointwise a vector with the largest $\ell^2(\mathbf{C}^d)$ magnitude is clearly wrong because the effect of weight $W(x)$ may depend strongly on the direction. In the special case where W is uniformly nonsingular (i.e. $\|W(x)\| \cdot \|W^{-1}(x)\| \leq C$ for all x) this can be controlled by a constant factor, but we have no such *a priori* assumptions about W.

For this reason our analysis will take place primarily in unweighted L^q spaces, following [4]. Rather than deal with T directly, we consider the action of $W^{1/p}TW^{-1/p}$ on functions in $L^q(dx)$. With the family of truncated operators $W^{1/p}T_\epsilon W^{-1/p}$ in mind, we define the maximal truncated operator $(W^{1/p}T)_*$ to be

$$(W^{1/p}T)_* f(x) = \sup_{\epsilon > 0} |W^{1/p}T_\epsilon f(x)| \tag{7}$$

with the convention that $f = W^{-1/p}g$ and g is a function in $L^q(dx)$. One estimate from the scalar theory that remains wholly intact is the bound

$$|W^{1/p}TW^{-1/p}g|(x) \leq |(W^{1/p}T)_* W^{-1/p}g|(x) + C|g(x)| \tag{8}$$

The constant C depends only on our choice of operator T but not on the function g. This will allow us to infer the boundedness of T by controlling the behavior of its truncations. Our primary results are the following four theorems, numbered according to the section in which they appear:

Four Theorems.

(**3.2**) *If W is a matrix A_p weight, there exists $\delta > 0$ such that the vector Hardy-Littlewood maximal function M_w (defined in section 3) is a bounded operator from $L^q(\mathbf{R}^n; \mathbf{C}^d)$ to $L^q(\mathbf{R}; \mathbf{R})$ whenever $|p - q| < \delta$.*

(**4.2**)*: Given a singular integral operator T as above, and a weight $W \in A_p$, there exists $\delta > 0$ such that $(W^{1/p}T)_* W^{-1/p}$ is a bounded operator from $L^q(\mathbf{R}^n; \mathbf{C}^d)$ to $L^q(\mathbf{R}; \mathbf{R})$ whenever $|p - q| < \delta$.*

(**5.1**)*: Consequently $W^{1/p}TW^{-1/p}$ is bounded on $L^q(\mathbf{R}^n; \mathbf{C}^d)$ for these exponents q.*

(**5.2**)*: In particular, T is bounded on $L^p(W)$ if $W \in A_p$. With one additional hypothesis on the structure of T, the converse statement is also true.*

Remark. The exponent $W^{1/p}$ is used throughout, even when we are considering functions under an L^q norm with $q \neq p$. This places us squarely in the setting of [25], where the A_p metric $W^{1/p}$ is the basic object of study. Theorem 5.1 then asserts that any A_p metric is also an A_q metric for all q in some open interval containing p.

2. Properties of A_p Weights

We would like first to characterize the matrix A_p class in a more transparent manner by borrowing a lemma from [12]:

Proposition 2.1. *A metric ρ_x satisfies the A_p condition if and only if the operators $f \to \chi_B \frac{1}{|B|}\int_B f\,dx$ are uniformly bounded on $L^p(\rho)$. In fact, the uniform bound is equal to the A_p constant of ρ.*

Proof. The $L^p(\rho)$ norm of $\chi_B \frac{1}{|B|}\int_B f\,dx$ is given by $\frac{1}{|B|}\big(\int_B \big[\rho_y\big(\int_B f\,dx\big)\big]^p dy\big)^{1/p}$, which in turn is equal to $|B|^{-1/p'}\rho_{p,B}\big(\int_B f\,dx\big)$. Therefore

$$\begin{aligned}\sup_{\|f\|_{L^p(\rho)}=1} \left\|\chi_B \tfrac{1}{|B|}\textstyle\int_B f\,dx\right\|_{L^p(\rho)} &= \sup_f \sup_{\mathbf{e}\in\mathbf{C}^d} |B|^{-1/p'} \frac{\int_B (\mathbf{e}, f(x))\,dx}{(\rho_{p,B})^*(\mathbf{e})} \\ &= \sup_{\mathbf{e}\in\mathbf{C}^d} |B|^{-1/p'} \frac{\|\chi_B \mathbf{e}\|_{L^{p'}(\rho^*)}}{(\rho_{p,B})^*(\mathbf{e})} = \sup_{\mathbf{e}\in\mathbf{C}^d} \frac{\rho^*_{p',B}(\mathbf{e})}{(\rho_{p,B})^*(\mathbf{e})}\end{aligned}$$

Equality between the first and second lines takes place because $L^p(\rho)$ is the dual space of $L^{p'}(\rho^*)$. ☐

Corollary 2.2. *Let ρ be an A_p metric. For any vector $\mathbf{v} \in \mathbf{C}^d$, $\rho_x(\mathbf{v})^p$ is a scalar A_p weight with constant less than or equal to that of ρ.*

Proof. Let ϕ be any scalar function and consider $f = \phi\mathbf{v}$. The weighted norm of f is $\|f\|_{L^p(\rho)} = (\int_B \phi^p\,[\rho_x(\mathbf{v})]^p dx)^{1/p}$. Proposition 2.1 applied to f states that all maps $\phi \to \chi_B \frac{1}{|B|}\int_B \phi\,dx$ are uniformly bounded on the L^p space with measure $[\rho_x(\mathbf{v})]^p dx$, with norms less than the A_p constant of ρ. We now apply Proposition 2.1 again, this time in the scalar setting, to conclude that $[\rho_x(\mathbf{v})]^p$ is a scalar A_p weight whose constant is also less than the A_p constant of ρ. ☐

Corollary 2.3. *If W is a matrix A_p weight, then $\|W\|$ is a scalar $A_p$weight.*

Proof. Let $\mathbf{e}_i$ be the standard unit basis for $\mathbf{C}^d$. Since $W(x)$ is a nonnegative and selfadjoint operator at each point x,

$$\begin{aligned}\|W(x)\| &= \|W^{2/p}(x)\|^{p/2} \sim [\mathrm{tr}(W^{2/p}(x))]^{p/2} \\ &= \Big(\sum_{i=1}^d |W^{1/p}(x)\mathbf{e}_i|^2\Big)^{p/2} \sim \sum_{i=1}^d |W^{1/p}(x)\mathbf{e}_i|^p\end{aligned} \tag{9}$$

pointwise in x. By corollary 2.2, each individual function $|W^{1/p}(x)\mathbf{e}_i|^p$ is a scalar A_p weight, therefore their sum is as well. ☐

Remarks. Both of these corollaries are proven in [23] for the case $p = 2$, and are adapted here with minimal alteration.

From this point forward we will work exclusively in the language of matrix weights. While our primary definition of A_p weights (6) is decidedly less elegant than that of A_p metrics (3), the ability to use notaion and theorems from linear algebra makes it a worthwhile sacrifice.

One crucial feature in the theory of scalar A_p weights is the presence of "Reverse Hölder" inequalities estimating the average of $W^{1+\epsilon}$ in terms of the average of W. We will employ inequalities of a similar character as the centerpiece of our analysis.

Proposition 2.4. *Let W be an A_p weight. Then there exist $\delta > 0$ and constants C_q such that for all balls $B \subset \mathbf{R}^n$,*

$$\frac{1}{|B|}\int_B \|W^{1/p}(y)V_B'\|^q dy \leq C_q, \text{ all } q < p+\delta \tag{10}$$

$$\frac{1}{|B|}\int_B \|V_B W^{-1/p}(y)\|^q dy \leq C_q, \text{ all } q < p'+\delta \tag{11}$$

Proof. We will verify only the first of these statements. The second one is proven in an identical manner with the starting point that $W^{-p'/p}$ is an $A_{p'}$ weight.

By Corollary 2.2, all functions of the form $|W^{1/p}(y)V_B'\mathbf{e}|^p$ are scalar A_p weights with A_p norms bounded uniformly in $\mathbf{e}$. It is therefore possible to choose q and C_q so that the Reverse Hölder inequality

$$\frac{1}{|B|}\int_B |W^{1/p}(y)V_B'\mathbf{e}|^q dy \;\leq\; C_q\Big(\frac{1}{|B|}\int_B |W^{1/p}(y)V_B'\mathbf{e}|^p dy\Big)^{q/p}$$

is satisfied for all $\mathbf{e} \in \mathbf{C}^d$.

Let $\mathbf{e}_i$ once again be the standard unit basis for $\mathbf{C}^d$. It is useful to remember that the norm of any $d\times d$ matrix M (not necessarily Hermitian) is controlled by its action on the vectors $\mathbf{e}_i$ via the formula

$$\|M\| \leq d^{1/2}\sup_i |M\mathbf{e}_i|$$

We may now estimate the desired integral:

$$\begin{aligned}\frac{1}{|B|}\int_B \|W^{1/p}(y)V_B'\|^q dy \;&\leq\; \frac{1}{|B|}\int_B \big(d^{1/2}\sup_i |W^{1/p}(y)V_B'\mathbf{e}_i|\big)^q dy \\ \leq d^{q/2}\sum_{i=1}^d \frac{1}{|B|}\int_B |W^{1/p}(y)V_B'\mathbf{e}_i|^q dy \;&\leq\; C_q\sum_{i=1}^d \Big(\frac{1}{|B|}\int_B |W^{1/p}(y)V_B'\mathbf{e}_i|^p dy\Big)^{q/p} \\ &\sim C_q\sum_{i=1}^d |V_BV_B'\mathbf{e}_i|^q \leq d\cdot C_q\|V_BV_B'\|^q \leq C_q.\end{aligned}$$

□

Note. In later sections we will also use the slightly weaker inequality

$$|B|^{-1}\int_B \|W^{1/p}(y)V_B^{-1}\|^q dy \leq C_q, \text{ all } q < p+\delta \tag{12}$$

whose proof follows the above calculations almost word for word.

3. The Hardy-Littlewood Maximal Function

There is a wide variety of possible maximal functions to choose from, each of which has its own advantages and limitations. In [4] we first considered an auxiliary maximal function M_w', given by

$$M_w'g(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |V_BW^{-1/p}(y)g(y)|\,dy \tag{13}$$

Although the intuitive meaning of M'_w is unclear, one may approach it with the classical tools of weak-type inequalities and interpolation. A direct application of the second reverse Hölder inequality (11) proves the following lemma.

Lemma 3.1. *Let W be an A_p weight. Then there exists $\delta > 0$ such that*

$$\|M'_w g\|_{L^q} \leq C_q \|g\|_{L^q(\mathbf{R}^n;\mathbf{C}^d)}, \ \ \textit{all}\ g \in L^q,\ \ \textit{all}\ q > p - \delta.$$

Sketch of Proof. The reverse Hölder inequality allows us to extend Proposition 2.1 to exponents $p - \delta < q < \infty$. For this maximal function one may use the Vitali Covering Lemma to obtain a weak-type (q, q) estimate. The result then follows from the Marcinkiewicz Interpolation Theorem.

The vector Hardy-Littlewood maximal function M_w is defined as

$$M_w g(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |W^{1/p}(x) W^{-1/p}(y) g(y)| dy \tag{14}$$

The following equivalent definition of M_w is often quite useful:

$$M_w g(x) = M\big(|W^{1/p}(x) W^{-1/p}(\cdot) g(\cdot)|\big)(x) \tag{15}$$

Here M denotes the classical Hardy-Littlewood maximal operator acting on scalar-valued functions. The only difference between M_w and M'_w is the presence of a weight $W^{1/p}(x)$ rather than an average weight V_B over a ball containing x. The reverse Hölder inequalities suggest that A_p weights are often pointwise comparable to their averages, in which case $\|M_w g\|$ would be controlled by $\|M'_w g\|$. For a range of exponents near p, this line of reasoning can be made precise.

Theorem 3.2. *Let W be an A_p weight. Then there exists $\delta > 0$ such that*

$$\|M_w g\|_{L^q} \leq C_q \|g\|_{L^q(\mathbf{R}^n;\mathbf{C}^d)}, \ \ \textit{all}\ g \in L^q,\ \ \textit{all}\ |p - q| < \delta.$$

Proof. Let us suppose for a moment that the suprema defining $M_w g$ and $M'_w g$ are taken over cubes in some dyadic grid. The entire preceding discussion holds for maximal functions over cubes, so in particular we can still estimate $\|M'_w g\|$ via Lemma 3.1. For each point x, choose a (dyadic) cube R_x such that

$$\begin{aligned} M_w g(x) &\leq 2|R_x|^{-1} \int_{R_x} |W^{1/p}(x) W^{-1/p}(y) g(y)| dy \\ &\leq 2\|W^{1/p}(x) V_{R_x}^{-1}\| \cdot \Big(|R_x|^{-1} \int_{R_x} |V_{R_x} W^{-1/p}(y) g(y)| dy\Big). \end{aligned}$$

For each integer j, define $\{S_j\}$ to be the collection of dyadic cubes $R = R_x$ that are maximal with respect to the property $2^j \leq |R|^{-1} \int_R |V_R W^{-1/p}(y) g(y)| dy < 2^{j+1}$. Maximality insures that whenever $M_w g(x) \neq 0$ the cube R_x is contained in some S_j with

$$|R_x|^{-1} \int_{R_x} |V_{R_x} W^{-1/p}(y) g(y)| dy \leq 2|S_j|^{-1} \int_{S_j} |V_{S_j} W^{-1/p}(y) g(y)| dy.$$

When j is fixed, the disjoint union $\cup_j S_j$ is contained in the set where $M'_w g(x) \geq 2^j$.

Consider the functions $N_Q(x) = \sup_{x\in R\subset Q} \|W^{1/p}(x)V_R^{-1}\|$, defined for $x \in Q$. By virtue of the preceding two statements, the inequality $M_w g(x) \le 4 \cdot 2^{j+1} N_{S_j}(x)$ must hold for some number j (this is trivial at the points where $M_w g(x) = 0$). It follows that

$$\|M_w g\|_{L^q}^q \le C \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{S_j} \int_{S_j} \big(N_{S_j}(x)\big)^q dx \tag{16}$$

By Lemma 3.3 below, we can continue the estimate as follows:

$$\|M_w g\|_{L^q}^q \le C \sum_{j=-\infty}^{\infty} 2^{jq} \sum_{S_j} |S_j| \le C \sum_{j=-\infty}^{\infty} 2^{jq} \left|\{M'_w g \ge 2^j\}\right| \le C \|M'_w g\|_{L^q}^q$$

The proof is then complete by Lemma 3.1.

□

Lemma 3.3. *Let W be a matrix A_p weight and functions $N_Q(x)$ be defined as above. Then there exist $\delta > 0$ and $C_q < \infty$ such that for all dyadic Q,*

$$\int_Q \big(N_Q(x)\big)^q dx \le C_q |Q| \text{ for all } q < p + \delta$$

Proof. We present an informal argument here, assuming that $\int_Q N_Q^q \le B|Q|$ with some finite B then deriving an *a priori* bound for B. This may be readily adapted into a rigorous proof.

Let $A < \infty$ be a large constant to be specified later. Denote by $\{R_j\}$ the set of maximal cubes satisfying $\|V_Q V_{R_j}^{-1}\| > A$. Outside of $\cup_j R_j$, $N_Q(x) \le A\|W^{1/p}(x)V_Q^{-1}\|$. Thus $\int_{Q\setminus\cup_j R_j} \big(N_Q(x)\big)^q dx \le C|Q|$, seen by applying reverse Hölder inequality (12).

We claim that $\sum_j |R_j| < \frac{1}{2}|Q|$ if A is sufficiently large. Remember first that $\|V_Q V_{R_j}^{-1}\| = \|V_{R_j}^{-1} V_Q\| \le C\|V'_{R_j} V_Q\|$, by Proposition 1.1. It follows that

$$\begin{aligned} |R_j| \cdot \|V'_{R_j} V_Q\|^{p'} &= \sup_{|\mathbf{e}|=1} |R_j| \cdot |V'_{R_j} V_Q \mathbf{e}|^{p'} \\ &\sim \sup_{|\mathbf{e}|=1} \int_{R_j} |W^{-1/p}(y) V_Q \mathbf{e}|^{p'} dy \le \int_{R_j} \|W^{-1/p}(y) V_Q\|^{p'} dy \end{aligned} \tag{17}$$

The cubes R_j are disjoint from one another, so

$$A^{p'} \sum_j |R_j| < C \int_{\cup_j R_j} \|W^{-1/p}(y) V_Q\|^{p'} dy \le C \int_Q \|W^{-1/p}(y) V_Q\|^{p'} \le C|Q|$$

This estimate shows that for A large enough, $\sum_j |R_j| < \frac{1}{2}|Q|$, and the value of A may be chosen independently of Q.

Inside each cube R_j, we may assume that $N_Q(x) = N_{R_j}(x)$, otherwise the bound $N_Q(x) \le A\|W^{1/p}(x)V_Q^{-1}\|$ still holds. Then

$$\int_{\cup_j R_j} \big(N_Q(x)\big)^q = \sum_j \int_{R_j} \big(N_{R_j}(x)\big)^q \le B \sum_j |R_j| < \frac{1}{2} B|Q|. \tag{18}$$

Putting these pieces together, we would discover that $B \leq C + \frac{1}{2}B$, where $C < \infty$ is determined by the constants in the reverse Hölder inequality. $\square$

This concludes the proof that matrix A_p weights enjoy L^q-boundedness of the dyadic Hardy-Littlewood maximal function for a range of exponents $|q-p| < \delta$. There is a standard argument employing two incompatible dyadic grids [7] for extending results of this kind to the general setting. Thus the Hardy-Littlewood maximal function as we originally defined it (as a supremum over balls containing x) is bounded in L^q for the same range of exponents q.

4. A Distributional Inequality

Proposition 4.1. *Let W be a matrix A_p weight and fix $q < 2 + \delta$. Then there exist positive constants $0 < b < 1, c > 0$ depending only on q, the A_p "norm" of W, and the dimensions d, n such that*

$$\Big|\big\{x \in \mathbf{R}^n : (W^{1/p}T)_*f(x) > \alpha; \max\big(M'_w(W^{1/p}f)(x), M_w(W^{1/p}f)(x)\big) < c\alpha\big\}\Big|$$

$$< \frac{1}{2}b^q\big|\{x \in \mathbf{R}^n : (W^{1/p}T)_*f(x) > b\alpha\}\big| \tag{19}$$

for all $f \in C_c^\infty(\mathbf{R}^n; \mathbf{C}^d)$

From this point onward we follow as closely as possible in the footsteps of Coifman and Fefferman [5], decomposing the set where $(W^{1/p}T)_*f > b\alpha$ into a union of cubes and proving the desired inequality on each cube separately. Our decomposition uses a slightly modified version of the Whitney covering lemma, stated below.

Covering Lemma. *Given a set $E \subset \mathbf{R^n}$ of finite (Lebesgue) measure, there exists a collection $\{Q_j\}$ of pairwise disjoint cubes such that:*

i) *$E \subset \cup_j Q_j$ up to sets of measure zero*

ii) $|Q_j \cap E| \geq \frac{1}{2}|Q_j|$

iii) $|3Q_j \cap E^c| \geq C_n|3Q_j|$

A simple consequence of statements i) and ii) is that $\sum_j |Q_j| \leq 2|E|$.

Proof. Let $\{Q_j\}$ be the collection of dyadic cubes maximal under the property that $|Q \cap E| \geq \frac{1}{2}|Q|$. Then conditions ii) and iii) hold with constant $C_n = \frac{1}{2} \cdot (\frac{2}{3})^n$. The first condition also holds because as $\epsilon \to 0$, the ratio $|B(x,\epsilon) \cap E|/|B(x,\epsilon)| \to 1$ at almost every $x \in E$. $\square$

Proof of Proposition 4.1. Write $f = W^{-1/p}g$ and let

$$E = \{x \in \mathbf{R}^n : (W^{1/p}T)_*f(x) > b\alpha\}$$

Apply the covering lemma to obtain cubes $\{Q_j\}$ with the specified properties. It suffices to verify that in each cube $Q = Q_j$ there is a distributional inequality

$$\big|\big\{x \in Q : (W^{1/p}T)_*f(x) > \alpha\,;\ \max\big(M'_w g(x), M_w g(x)\big) < c\alpha\big\}\big| < \frac{1}{4}b^q|Q|. \tag{20}$$

For this we use a construction similar to the one in [5]. Let O be the ball with the same center as Q_j and radius 5 diam (Q). By the covering lemma and inequality (11), there exists a point $\overline{x} \in 3Q$ such that

$$(W^{1/p}T)_* f(\overline{x}) < b\alpha \text{ and } \|V_O W^{-1/p}(\overline{x})\| < C$$

Let $B = B\big(\overline{x}, 3\ \mathrm{diam}(Q_j)\big)$. Since $B \subset O$ and is of comparable size, $\|V_B V_O^{-1}\|$ is bounded by a constant and hence $\|V_B W^{-1/p}(\overline{x})\| < C$.

Assume $\big|\{x \in Q : M'_w g(x) < c\alpha\}\big| \geq \frac{1}{4} b^q |Q|$, otherwise the proposition is trivially satisfied. Then there exists a point $\overline{y} \in Q$ such that

$$M'_w g(\overline{y}) < c\alpha \text{ and } \|V_B W^{-1/p}(\overline{y})\| \leq C b^{-1}$$

Write $f_1 = \chi_B f$ and $f_2 = \chi_{B^c} f$. By the sublinearity of $(W^{1/p}T)_*$, the set where $(W^{1/p}T)_* f(x) > \alpha$ is contained in the union of sets $(W^{1/p}T)_* f_i(x) > \alpha/2,\ i = 1, 2$.

The operator T_* is weak-type $(1, 1)$. This fact is easily obtained from the scalar case when d is finite, but is also true in general [17]. Consequently,

$$|\{(V_B T)_* f_1(x) > \frac{\alpha}{2R}\}| \leq \frac{AR}{\alpha} \|V_B f_1\|_{L_1(\mathbf{R}^n; \mathbf{C}^d)}$$

Here we are using the property that operator T_* commutes with multiplication by any constant matrix, in this case V_B. Furthermore,

$$\|V_B f_1\|_{L^1} = \int_B |V_B f(y)| dy \leq |B| M'_w g(\overline{y}) \leq Cc\alpha |Q|$$

with the end result that $\big|\{x \in Q : (V_B T)_* f_1(x) > \frac{\alpha}{2R}\}\big| \leq CcR|Q|$.

It follows that $|\{x \in Q : (W^{1/p}T)_* f_1(x) > \frac{\alpha}{2}\}| \leq (CcR + C'R^{-p})|Q|$ for all $R > 0$, because the Reverse Hölder inequality (10) guarantees that $\|W^{1/p}(x) V_B^{-1}\| < R$ except on a set of measure less than $C'R^{-p}$. Taking the infimum over R,

$$\big|\{x \in Q : (W^{1/p}T)_* f_1(x) > \alpha/2\}\big| \leq C_0 c^{p/(p+1)} |Q| \tag{21}$$

For the second estimate, we begin by noting that the point $\overline{x}$ is chosen so that $(W^{1/p}T)_* f(\overline{x}) < b\alpha$ and $\|V_B W^{-1/p}(\overline{x})\| < C$. Then $(V_B T)_* f(\overline{x}) < Cb\alpha$. Our estimate for $|\{(W^{1/p}T)_* f_2(x) > \alpha/2\}|$ relies on the following inequality which holds for all $x \in Q$.

$$\begin{aligned}(V_B T)_* f_2(x) &\leq (V_B T)_* f(\overline{x}) + C' M\big(|V_B f|\big)(\overline{y}) \\ &\leq Cb\alpha + C' \|V_B W^{-1/p}(\overline{y})\| \cdot M\big(|W^{1/p} f|\big)(\overline{y}) \\ &\leq Cb\alpha + C' \|V_B W^{-1/p}(\overline{y})\| \cdot M_w g(\overline{y}) \leq (Cb + C' b^{-1} c)\alpha\end{aligned} \tag{22}$$

In the preceding expressions $M(\cdot)$ denotes the scalar Hardy-Littlewood maximal function.

Imitating the method for the $|(W^{1/p}T)_* f_1|$ estimate, we see that

$$\big|\{x \in Q : (W^{1/p}T)_* f_2(x) > R(Cb + C' b^{-1} c)\alpha\}\big| \leq AR^{-r} |Q|$$

where r may be chosen so that $q < r < p+\delta$. Once again (10) has been invoked, this time to guarantee that $\|W^{1/p}V_B^{-1}\| > R$ only on a set of measure less than $CR^{-r}|B|$. Set R equal to $(4bC)^{-1}$. Then

$$\left|\{x \in Q : (W^{1/p}T)_*f_2(x) > (1/4 + C_1 b^{-2} c)\alpha\}\right| \leq C_2 b^r |Q| \tag{23}$$

Statement (20) is then verified by choosing $b < (8C_2)^{1/q-r}$ and c sufficiently small. Summing over all cubes Q_j proves the proposition. □

Corollary 4.2. *With c as in Proposition* (4.1)*,*

$$\left\|(W^{1/p}T)_*f\right\|_{L^q}^q \leq 2c^{-q}\left\|\max\bigl(M'_w(W^{1/p}f), M_w(W^{1/p}f)\bigr)\right\|_{L^q}^q$$

for all $f \in C_c^\infty(\mathbf{R}^n;\mathbf{C}^d)$

Proof. If both sides of (19) are multiplied by $q\alpha^{q-1}$ and integrated over the the interval $0 \leq \alpha < \infty$, the resulting inequality is

$$\int_{\mathbf{R}^n} \Bigl([(W^{1/p}T)_*f]^q - c^{-q}\max\bigl([M'_w(W^{1/p}f)]^q, [M_w(W^{1/p}f)]^q\bigr)\Bigr)_+ dx$$
$$\leq \frac{1}{2}\int_{\mathbf{R}^n} [(W^{1/p}T)_*f]^q\, dx$$

from which it follows that

$$\|(W^{1/p}T)_*f\|_{L^q}^q - \frac{1}{c^q}\left\|\max\bigl(M'_w(W^{1/p}f), M_w(W^{1/p}f)\bigr)\right\|_{L^q}^q \leq \frac{1}{2}\|(W^{1/p}T)_*f\|_{L^q}^q$$

The remaining task is to verify that the L^q norm of $(W^{1/p}T)_*f$ is finite. A key estimate is the fact that $T_*f(x) \leq C_f(1+|x|)^{-n}$ for all $f \in C_c^\infty$, where C_f depends on f. Then

$$(W^{1/p}T)_*f(x) \leq C\|W\|^{1/p}(1+|x|)^{-n}$$

There are many ways to show that the expression on the right-hand side is in L^q, all exploiting the fact that $\|W\|$ is a scalar A_p weight. One possibility is to choose any nontrivial (scalar) function $\phi \geq 0 \in C_c^\infty$. We have shown in Theorem 3.2 that $\|W\|^{1/p}M(\|W\|^{-1/p}\phi) \in L^q$ whenever $|p-q| < \delta$.

On the other hand, $C(1+|x|)^{-n} \leq M(\|W\|^{-1/p}\phi)$, which completes the proof. □

5. The Main Theorem

Theorem 5.1. *Let T be a linear operator whose associated convolution kernel $K(x)$ satisfies the hypotheses in* (1)*, and which acts separately on each coordinate function of f (in other words, $(Tf)_j = Tf_j$). Let W be a matrix A_p weight.*

There exists $\delta > 0$ such that $W^{1/p}TW^{-1/p}$ is a bounded operator on $L^q(\mathbf{R}^n;\mathbf{C}^d)$ whenever $|q-p| < \delta$.

Proof. As in the scalar case, the truncated operators T_ϵ possess a weak limit T_0, and $T = T_0 + A$, where A is a bounded pointwise multiplier. In dimensions $d > 1$, $A = A(x)$ is a matrix-valued function, but the hypothesis $\Lambda T\Lambda^{-1} = T$ requires $A(x)$ to be a scalar L^∞ function multiplied by the identity matrix.

The function $W^{1/p}TW^{-1/p}g$ is dominated pointwise by g and $(W^{1/p}T)_*(W^{-1/p}g)$, as in equation (8):

$$\begin{aligned}|W^{1/p}TW^{-1/p}g(x)| &= |W^{1/p}T_0W^{-1/p}g(x) + A(x)g(x)| \\ &\leq |(W^{1/p}T)_*(W^{-1/p}g)(x)| + C|g(x)|.\end{aligned}$$

The triangle inequality for L^q-norms immediately yields the result

$$\left\|W^{1/p}TW^{-1/p}g\right\|_{L^q} \leq \left\|(W^{1/p}T)_*W^{-1/p}g\right\|_{L^q} + C\|g\|_{L^q} \tag{24}$$

For all g such that $W^{-1/p}g \in C_c^\infty$, the right-hand side is controlled by $\|g\|_{L^q}$. Observe that $W^{q/p}$ is a locally integrable matrix-valued function. Then $C_c^\infty(\mathbf{R}^n;\mathbf{C}^d)$ is a dense subset of $L^q(W^{q/p})$. The map $f \in L^q(W^{q/p}) \to g = W^{1/p}f \in L^q(dx)$ is an invertible isometry, so its image $W^{1/p}(C_c^\infty)$ is dense in L^q. Thus the boundedness of $W^{1/p}TW^{-1/p}$ may then be extended to all functions $g \in L^q(\mathbf{R}^n;\mathbf{C}^d)$, $|p-q| < \delta$. $\square$

A converse statement, with some minor modifications, is also true.

Theorem 5.2. *Suppose that T is a convolution operator as above, with the additional nondegeneracy hypothesis that there exists some unit vector $\mathbf{u} \in \mathbf{R}^n$ such that $|K(r\mathbf{u})| \geq a|r|^{-n}$, all $r \in \mathbf{R} \setminus \{0\}$. If T is a bounded operator on $L^p(W)$, then W is an A_p weight.*

In order to prove this theorem we first need a result about integral operators with bounded and compactly supported kernels:

Proposition 5.3. *Let S be an integral operator $Sf(x) = \int_{\mathbf{R}^n} S(x,y)f(y)$ whose (scalar) kernel $S(x,y)$ is supported in $B \times B$ and satisfies the bound $|S(x,y)| \leq |B|^{-1}$ for all $(x,y) \in B \times B$.*

The norm of S as an operator on $L^p(W)$ is less than $C_d\|V_BV'_B\|$, where C_d is a dimensional constant independent of the particular choice of S. In the special case $S_0(x,y) = |B|^{-1}\chi_{B\times B}$, the operator norm of S_0 is also greater than $C_d^{-1}\|V_BV'_B\|$.

Proof. This is a straightforward calculation similar to those found in Section 2. Let f be any function in $L^p(W)$. We first estimate the size of $W^{1/p}(x)Sf(x)$ pointwise for each x.

$$\begin{aligned}|W^{1/p}(x)Sf(x)| &= \Big|W^{1/p}(x)\int_B S(x,y)f(y)\,dy\Big| \\ &= \Big|\int_B S(x,y)W^{1/p}(x)f(y)\,dy\Big| \leq |B|^{-1}\int_B |W^{1/p}(x)f(y)|dy \\ &\leq |B|^{-1}\Big(\int_B \left\|W^{1/p}(x)W^{-1/p}(y)\right\|^{p'}dy\Big)^{1/p'} \cdot \|f\|_{L^p(W)}.\end{aligned}$$

As in Section 2, we now introduce an orthonormal basis of vectors $\mathbf{e}_i$ spanning $\mathbf{C}^d$.

$$\Big(\int_B \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}dy\Big)^{1/p'}$$
$$\leq \Big(\int_B \big(d^{1/2}\sup_i |W^{-1/p}(y)W^{1/p}(x)\mathbf{e}_i|\big)^{p'}dy\Big)^{1/p'}$$
$$\leq d^{1/2}\Big(\sum_{i=1}^d \int_B |W^{-1/p}(y)W^{1/p}(x)\mathbf{e}_i|^{p'}dy\Big)^{1/p'}$$
$$\leq C_d\Big(\sum_{i=1}^d |B|\cdot|V'_BW^{1/p}(x)\mathbf{e}_i|^{p'}\Big)^{1/p'} \leq C_d|B|^{1/p'}\|V'_BW^{1/p}(x)\|$$

which leads to the estimate $|W^{1/p}(x)Sf(x)| \leq C_d|B|^{-1/p}\|V'_BW^{1/p}(x)\|\cdot\|f\|_{L^p(W)}$.

Then for all $\|f\|_{L^p(W)} \leq 1$, it follows that

$$\|Sf\|_{L^p(W)} \leq C\Big(|B|^{-1}\int_B \|V'_BW^{1/p}(x)\|^p dx\Big)^{1/p} \tag{25}$$
$$\leq C_d\Big(|B|^{-1}\int_B \big(d^{1/2}\sup_i |W^{1/p}(x)V'_B\mathbf{e}_i|\big)^p\Big)^{1/p}$$
$$\leq C_d\Big(\sum_i |B|^{-1}\int_B |W^{1/p}(x)V'_B\mathbf{e}_i|^p\Big)^{1/p}$$
$$\sim C_d\Big(\sum_i |V_BV'_B\mathbf{e}_i|^p\Big)^{1/p} \leq C_d\|V_BV'_B\|$$

The second assertion is a restatement of Proposition 2.1. □

Proof of Theorem 5.2. First, let $\epsilon > 0$ be small enough so that $2\epsilon+\epsilon^2 < \frac{1}{2}C_d^{-2}$. There exists a number $t_0 < \infty$ such that

$$|K(\mathbf{v}) - K(rt_0\mathbf{u})| \leq \epsilon|K(rt_0\mathbf{u})| \text{ whenever } \mathbf{v} \in B(rt_0\mathbf{u}, 2r), \text{ all } r \in \mathbf{R}\setminus\{0\}. \tag{26}$$

This is seen to be true because $|K(rt_0\mathbf{u})| \geq \frac{a}{t_0^n|r|^n}$ but $|\nabla K(x)| \leq \frac{C}{t_0^{n+1}r^{n+1}}$ for all $x \in B(rt_0\mathbf{u}, r)$. It suffices to choose $t_0 > \frac{2C}{\epsilon a}$.

Let B denote the ball $B(y, r)$ in $\mathbf{R}^n$, and B' the translated ball $B' = B(y+rt_0\mathbf{u}, r)$. We wish to consider the operator S_B defined by

$$S_Bf = \chi_BT\big(\chi_{B'}T(\chi_Bf)\big)$$

This is an integral operator whose kernel $S_B(x,y) = \chi_{B\times B}\int_{B'} K(x-z)K(z-y)\,dz$ is supported in $B\times B$. If T acts boundedly on $L^p(W)$, so too does S_B with operator norm less than or equal to $\|T\|^2$.

The restrictions $\{x, y \in B, z \in B'\}$ guarantee that $z-y \in B(rt_0\mathbf{u}, 2r)$ and $x-z \in B(-rt_o\mathbf{u}, 2r)$. Thus the values of $K(z-y)$ and $K(x-z)$ do not vary much over the region of integration. Using the bounds established in (26), we rewrite $S_B(x,y)$ as the sum of a characteristic function and a small remainder:

$$S_B(x,y) = |B|K(rt_0\mathbf{u})K(-rt_0\mathbf{u})\chi_{B\times B} + S_1(x,y), \tag{27}$$

where $|S_1(x,y)| \le \frac{1}{2}C_d^{-2}|B| \cdot |K(rt_0\mathbf{u})K(-rt_0\mathbf{u})|$

According to Proposition 5.3, the first term corresponds to an operator with norm at least $C\|V_B V_B'\|$. In terms of other constants, C is proportional to $a^2 t_0^{-2n} C_d^{-1}$. The operator corresponding to the second term has norm no more than half as great. It follows that $\|S_B\| \ge \frac{1}{2}C\|V_B V_B'\|$. Then

$$\|V_B V_B'\| \le 2C^{-1}\|S_B\| \le 2C^{-1}\|T\|^2 < \infty \tag{28}$$

for all balls $B \subset \mathbf{R}^n$, and W is an A_p weight. □

Corollary 5.4. *If W is a matrix A_p weight, there exists $\delta > 0$ such that $W^{q/p}$ is an A_q weight whenever $|q-p| < \delta$. In other words, an A_p metric is also an A_q metric for all $|q-p| < \delta$.*

Remarks. We could have proven this statement directly in section 2, using the reverse Hölder inequality to show that operators $f \to \chi_B \frac{1}{|B|}\int_B f\,dx$ are uniformly bounded on $L^q(W^{q/p})$. To do so would have added another computation without simplifying the subsequent discussion in any way.

Recall that a matrix weight $W \in A_p$ if and only if the averaging operators A_B defined by

$$A_B f = \chi_B \frac{1}{|B|}\int_B f\,dx$$

are uniformly bounded on $L^p(W)$. An equivalent statement is that the conjugated operators $W^{1/p}A_B W^{-1/p}$ are uniformly bounded on the unweighted space $L^p(\mathbf{C}^d)$. It is trivial to observe that A_B are uniformly bounded on $L^\infty(\mathbf{C}^d)$ with norm 1. By interpolation on the analytic family of operators[1]

$$\left\{W^{(1-z)/p}A_B W^{(z-1)/p}, \quad 0 \le \mathrm{Re}(z) \le 1\right\}$$

we find that $W^{1/r}A_B W^{-1/r}$ are uniformly bounded on $L^r(dx)$ for all $r > p$, leading to another result well known in the scalar case:

Proposition 5.5. *If W is a matrix A_p weight, then W is also a matrix A_r weight for all $r > p$.*

One crucial difference must be noted. We cannot use the reverse Hölder inequality in this setting to extend the range of exponents to $r > p - \delta$. If we could, then by corollary 5.4 and proposition 5.5 for each weight $W \in A_p$ there would exist numbers $r < q < p$ such that $W^{q/r} \in A_q \subset A_p$. Instead, counterexamples are known; in [1] a matrix A_2 weight W is constructed for which $W^s \notin A_2$ for any $s > 1$.

On a speculative note, perhaps this (suspected) lack of self-improvement is related to the absence of a unifying matrix A_∞ class whose elements are all contained in some A_p with p finite. We do not claim to have proven anything here, nor have we investigated thoroughly the union of the A_p-weight classes in search of a common A_∞ property. It has been suggested [25] that the scalar A_∞ condition generalizes instead

[1]Following [16], with the slight modification $F_z(\psi) = |\psi|^{\frac{\alpha(z)}{\alpha}-1}\psi$

to an entire spectrum of $A_{p,\infty}$ conditions, one for each exponent p, in the matrix setting.

6. The Case $d = \infty$

Most of the estimates in the preceding discussion fail when the dimension d is infinite. Banach space norms may not be representable by matrices, and traces (when defined at all) are no longer comparable to operator norms. Most importantly, the main theorem is false. Gillespie et al. [9] have constructed operator A_2 weights W for which the Hilbert Transform is unbounded on $L^2(W)$.

The test function f in their counterexample is constructed out of Haar functions on different length-scales, with the signs chosen so that each new piece contributes positively to the overall $L^2(W)$ norm of Tf. Linearity of T is needed to ensure that the whole of Tf will be equal to the sum of the various parts, and also to bound from below an expectation over choices of signs. When applied to merely sublinear operators such as a maximal function, the argument is less successful. We do not presently know if the Hardy-Littlewood maximal operator M_w is bounded or not.

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