DIAGONAL FORMS AND ZERO-SUM (MOD 2) BIPARTITE RAMSEY NUMBERS

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ABSTRACT. Let $G$ be a subgraph of a complete bipartite graph $K_{n,n}$. Let $N(G)$ be a 0-1 incidence matrix with edges of $K_{n,n}$ against images of $G$ under the automorphism group of $K_{n,n}$. A diagonal form of $N(G)$ is found for every $G$, and whether the row space of $N(G)$ over $\mathbb{Z}_2$ contains the vector of all 1’s is determined. This re-proves Caro and Yuster’s results on zero-sum bipartite Ramsey numbers [3], and provides necessary and sufficient conditions for the existence of a signed bipartite graph design.

1. Introduction

Let $G$ be a nonempty subgraph of the complete bipartite graph $K_{n,n}$ with $2n$ vertices, some of which are possibly isolated. Let $h$ be the characteristic vector of $G$, i.e. $h$ is a column vector of length $n^2$ indexed by the edges of $K_{n,n}$, with 1 if the edge is in $G$ and 0 otherwise. Let $\Gamma$ be the automorphism group on the vertices of $K_{n,n}$. Let $N = N(G)$ be the matrix with $2(n!)^2$ columns, each column representing an image of $h$ under the action of $\Gamma$ on the vertices.

For any integer matrix $A$, there exist integer square matrices $E$ and $F$ with determinants $\pm 1$ such that $EAF = D$ is a diagonal matrix, i.e. the $(i,j)$-entry of $D$ is 0 unless $i = j$. This $D$ is called a diagonal form of $A$. If all diagonal entries $d_1, d_2, \ldots$ are non-negative and $d_i$ divides $d_{i+1}$ for all $i$, then $D$ is called the Smith normal form of $A$ which is unique with respect to $A$.

This paper is going to give the diagonal forms of $N(G)$ for all $G$’s (see Theorems 5 and 12), which can be used to re-produce the results on zero-sum (mod 2) bipartite Ramsey numbers given in [3], as well as giving necessary and sufficient conditions for the existence of a signed bipartite graph design. Some techniques in this paper are introduced in [8] and [9].

2. Diagonal Forms of $U \cdot N(G)$

Two integer matrices $A$ and $B$ of the same size are $\mathbb{Z}$-equivalent if $B$ can be obtained from $A$ by a sequence of integral row and column operations (adding an integer multiple of one row or column to another row or column, or multiplying a row or column by $-1$). Alternatively, $A$ and $B$ are $\mathbb{Z}$-equivalent if there exist integer square matrices $E$ and $F$ with determinants $\pm 1$ such that $EAF = B$. If $A$ is $\mathbb{Z}$-equivalent to a diagonal matrix $D$, i.e. $EAF = D$ where $E$ and $F$ are integer square matrices with determinants $\pm 1$, then $D$ is called a diagonal form of $A$, $E$ a front and $F$ a back of $A$. We will call the set of diagonal entries of $D$ as a set of diagonal factors of $A$.

If $A$ is $\mathbb{Z}$-equivalent to an identity matrix, then $A$ is said to be unimodular. In fact, it is easy to see that a square integer matrix is unimodular if and only if its determinant is $\pm 1$. If there

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is a submatrix $A'$ of $A$ by deleting some columns such that $A'$ is $\mathbb{Z}$-equivalent to an identity matrix, then $A$ is said to be row-unimodular. Any row-unimodular $A$ has a unimodular extension $\tilde{A}$, which is an extension of $A$ by adding rows below $A$ and is unimodular.

Let $W$ be a $2n \times n^2$ incidence matrix of $K_{n,n}$ with vertices against edges. Let $U$ be a $(2n-1) \times n^2$ matrix obtained from $W$ with the first row replaced by $1$, the vector of all ones, and the last row deleted. The matrix $U$ is row-unimodular since $U$ has a submatrix

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<th>$1_{n-1}$</th>
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<tbody>
<tr>
<td>$O_{(n-1)\times(n-1)}$</td>
<td>$0_{n-1}^\top$</td>
<td>$I_{(n-1)\times(n-1)}$</td>
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<tr>
<td>$I_{(n-1)\times(n-1)}$</td>
<td>$0_{n-1}^\top$</td>
<td>$O_{(n-2)\times(n-1)}$</td>
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which is $\mathbb{Z}$-equivalent to $I_{(2n-1)\times(2n-1)}$. Here, and throughout the paper, $1_i$ and $0_i$ always denote row vectors of all ones and all zeros respectively of length $i$, and $O_{i\times j}$ denotes a zero matrix of dimensions $i \times j$.

Let $G$ be a nonempty subgraph of the complete bipartite graph $K_{n,n}$ with degrees $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, where $a_i$'s are the degrees of the vertices in one partite set and $b_i$'s in the other one, and some of them are possibly zeroes. Let $h$ be the characteristic vector of $G$, i.e. $h$ is a column vector of length $n^2$ indexed by the edges of $K_{n,n}$, with 1 if the edge is in $G$ and 0 otherwise. Let $\Gamma \cong S_n \wr \{a,b\} \cong S_2$ be the permutation group on the vertices of $K_{n,n}$ which can permute vertices within each partite set and interchange the two partite sets. Let $N = N(G)$ be the matrix with $2(n!)^2$ columns, each column representing an image of $h$ under the action of $\Gamma$ on the vertices.

In $U \cdot N(G)$, each column is either $(m, a_{i_1}, a_{i_2}, \ldots, a_{i_n}, b_{j_1}, b_{j_2}, \ldots, b_{j_n})^\top$ or $(m, b_{i_2}, b_{i_3}, \ldots, b_{i_n}, a_{j_2}, a_{j_3}, \ldots, a_{j_n})^\top$, where $m$ is the number of edges of $G$ and \( \{i_1, i_2, \ldots, i_n\} = \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}. \) Pick two columns in $UN$ that are identical except one entry, e.g. $(m, a_1, a_3, \ldots, a_n, b_1, \ldots, b_{n-1})^\top$ and $(m, a_2, a_3, \ldots, a_n, b_1, \ldots, b_{n-1})^\top$. Taking the difference of these two columns, we get $(0, a_1 - a_2, 0, \ldots, 0)^\top$ in the column module of $UN$ over $\mathbb{Z}$. Hence, the column module of $UN$ contains $ge_k^\top$, $k = 2, \ldots, 2n - 1$, where $g = \gcd\{a_i - a_j, b_i - b_j\}$ over $1 \leq i, j \leq n$ and $e_k$ denotes the $k$-th standard unit vector of length $2n - 1$. So the matrix

<table>
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<tr>
<th>$m$</th>
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<th>$0_{2n-2}$</th>
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<tbody>
<tr>
<td>$a_1 1_{n-1}^\top$</td>
<td>$b_1 1_{n-1}^\top$</td>
<td>$gl_{(2n-2)\times(2n-2)}$</td>
</tr>
<tr>
<td>$b_1 1_{n-1}^\top$</td>
<td>$a_1 1_{n-1}^\top$</td>
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has the same column module as \( UN \). After some integral row and column operations, we can see that the matrix

\[
\begin{array}{c|c|c}
 m & 0 & 0_{2n-3} \\
 a_1 & \tilde{g} & 0_{2n-3} \\
 0_{n-2}^\top & 0_{n-2}^\top & gI_{(2n-3)\times(2n-3)} \\
 (a_1 + b_1)1_{n-1}^\top & 0_{n-1}^\top & \\
\end{array}
\]

has the same diagonal form as \( UN \), where

\[ \tilde{g} = \gcd\{a_i - b_j, a_i - a_j, b_i - b_j\} = \gcd\{a_1 - b_1, g\}. \]

By computing the determinantal divisors (see [6] for details), we have the following theorem.

**Theorem 1.** A set of diagonal factors of \( UN \) is

\[
\left(\frac{\ell \alpha}{\ell \alpha}\right)^1, (\tilde{g}c)^1, \left(\frac{h}{\ell}\right)^{2n-4},
\]

where \( h = \gcd\{a_i, b_j\} = \gcd\{a_1, \tilde{g}\} \) and \( c = \gcd\left\{\frac{m}{h}, \frac{a_1 + b_1}{\tilde{g}}, \frac{g}{\tilde{g}}\right\} \). Here, \((x)^y\) means the diagonal entry \( x \) occurs with multiplicity \( y \). (Note that \( h \neq 0 \) since \( G \) is nonempty. Note also that \( \tilde{g} = 0 \) if and only if \( g = 0 \), and in this case, we define \( \tilde{g} = 1 \).)

**Proof.** The gcd of the determinants of \( 1 \times 1 \) submatrices in (1) is \( \gcd\{a_1, \tilde{g}\} = h \).

The gcd of the determinants of \( i \times i \) submatrices in (1), \( 2 \leq i \leq 2n-2 \), is \( \gcd\{m\tilde{g}^i, \tilde{g}^{i-1}, a_1\tilde{g}^{i-1}, \tilde{g}^{i-2}(a_1 + b_1)\} = \tilde{g}g^{i-2}hc \).

The determinant of the full matrix in (1) is \( m\tilde{g}2^{2n-3} \).

**Lemma 2.** For any \( x, y \in \mathbb{Z} \), there exists \( a, b \in \mathbb{Z} \) such that \( \gcd\{a, x, y\} = 1 \) and \( ax + by = \gcd\{x, y\} \).

**Proof.** Let \( \gcd\{x, y\} = d \), \( x = x'd \) and \( y = y'd \). Let \( a_0, b_0 \in \mathbb{Z} \) be such that \( a_0x' + b_0y' = 1 \). Note that \( \gcd\{a_0, y'\} = 1 \). Our goal is to find \( a \) such that \( a \equiv a_0 \pmod{y'} \) and \( \gcd\{a, d\} = 1 \).

Without loss of generality, we can assume that all prime factors of \( d \) are of degree 1. Let \( d = d_1d_2 \) such that \( d_1 \mid y' \) and \( \gcd\{d_2, y'\} = 1 \). As \( \gcd\{a_0, y'\} = 1 \), it suffices to find \( a \) such that \( a \equiv a_0 \pmod{y'} \) and \( a \equiv 1 \pmod{d_2} \), which is possible by the Chinese remainder theorem.

**Theorem 3.** By Lemma 2, let \( \alpha, \sigma \in \mathbb{Z} \) be such that \( \gcd\left\{\alpha, \frac{a_1 + b_1}{h}, \frac{g}{\tilde{g}}\right\} = 1 \) and \( -\alpha \frac{a_1 + b_1}{h} + \sigma \frac{g}{\tilde{g}} = \gcd\left\{\frac{a_1 + b_1}{h}, \frac{g}{\tilde{g}}\right\} \). Then there exist \( \ell, \ell' \in \mathbb{Z} \) such that \( \ell' \gcd\left\{\frac{a_1 + b_1}{h}, \frac{g}{\tilde{g}}\right\} \equiv 0 \pmod{\ell} \). Let \( \beta, \tau \in \mathbb{Z} \) be such that \( -\beta a_1 + \tau \tilde{g} = h \). Let \( \ell'' = \beta \ell + (a_1 + b_1) \). A front \( E \) of \( UN \) is

\[
\begin{array}{c|c|c|c|c}
\ell \gcd\left\{\frac{a_1 + b_1}{h}, \frac{g}{\tilde{g}}\right\} & \alpha \frac{\ell \alpha}{\ell \alpha} + \sigma \frac{\ell' \ell''}{\ell' \ell''} & 0_{n-2} & \alpha \frac{\ell \alpha}{\ell \alpha} & 0_{n-2} \\
\ell' + \ell'' & 0_{n-2} & \ell' & 0_{n-2} \\
0 & 1 & 0_{n-2} & 0_{n-2} \\
0 -1^\top_{n-2} & I_{(n-2)\times(n-2)} & 0^\top_{n-2} & O_{(n-2)\times(n-2)} \\
0^\top_{n-2} & 0^\top_{n-2} & O_{(n-2)\times(n-2)} & -1^\top_{n-2} & I_{(n-2)\times(n-2)} \\
\end{array}
\]
where the first three rows correspond to the diagonal entries \( \frac{mg}{hc}, \tilde{g}c \) and \( h \) respectively, and the other rows correspond to the diagonal entries \( g \).

Proof. We first show that \( E \) is unimodular. Given that \( \gcd \{ \alpha, \frac{a_1 + b_1}{h}, \frac{g}{g} \} = 1 \), we have \( \gcd \{ \frac{m}{h}, \frac{a_1 + b_1}{h}, \frac{g}{g} \} = c \), or \( \gcd \{ \frac{m}{h}, \frac{1}{c} \gcd \{ \frac{a_1 + b_1}{h}, \frac{g}{g} \} \} = 1 \). Hence, there exist \( \ell, \ell' \in \mathbb{Z} \), \( \gcd \{ \ell, \ell' \} = 1 \), such that \( \frac{\ell}{c} \gcd \{ \frac{a_1 + b_1}{h}, \frac{g}{g} \} - \ell' \frac{m}{hc} = 1 \). So the submatrix

\[
\begin{array}{ccc}
\frac{\ell}{c} \gcd \{ \frac{a_1 + b_1}{h}, \frac{g}{g} \} & \frac{m}{h} + \sigma \frac{mg}{hc} & \frac{m}{hc} \\
\ell & \ell' + \tilde{g} & \tilde{g}' \\
0 & 1 & 0
\end{array}
\]

has determinant \(-1\), which implies \( E \) is unimodular.

From (1), we note that the column module of \( UN \) is the same as that of

\[
M = \begin{pmatrix}
m & 0 & 0_{2n-3} \\
a_11_{n-1}^\top & \tilde{g}1_{n-1}^\top & 0_{2n-3} \\
b_11_{n-1}^\top & -\tilde{g}1_{n-1}^\top & gI_{(2n-3)\times(2n-3)}
\end{pmatrix}.
\]

The product of the first row of \( E \) with the first column of \( M \) is

\[
\frac{m}{c} \left( \gcd \{ \frac{a_1 + b_1}{h}, \frac{g}{g} \} + \alpha \frac{a_1 + b_1}{h} + \sigma \frac{a_1 + b_1}{h} \right) = \frac{m}{c} \left( -\alpha \frac{a_1 + b_1}{h} + \frac{g}{g} + \alpha \frac{a_1 + b_1}{h} + \sigma \frac{a_1 + b_1}{h} \right) = \sigma \frac{mg}{hc}(h + \beta a_1) = \sigma \frac{mg}{hc}(-\beta a_1 + \tau \tilde{g} + \beta a_1) = \sigma \tau \frac{mg}{hc}.
\]

The product of the first row of \( E \) with the second column of \( M \) is \( \sigma \beta \frac{mg}{hc} \), and the product of the first row of \( E \) with the \((n + 1)\)-th column of \( M \) is \( \alpha \frac{mg}{hc} \). From the definition, it is clear that \( \gcd \{ \beta, \tau \} = \gcd \{ \alpha, \sigma \} = 1 \), so \( \gcd \{ \sigma, \tau, \beta, \alpha \} = 1 \), and the first row of \( E \) corresponds to the diagonal entry \( \frac{mg}{hc} \).

The product of the second row of \( E \) with the first column of \( M \) is

\[
\ell m + \ell' (a_1 + b_1) + \ell'' a_1 = \frac{\ell m + \ell' (a_1 + b_1)}{h} (h + \beta a_1) = \frac{\ell m + \ell' (a_1 + b_1)}{hc} \tau \tilde{g}c.
\]

The product of the second row of \( E \) with the second column of \( M \) is \( \ell'' \tilde{g} = \beta \frac{\ell m + \ell' (a_1 + b_1)}{hc} \tilde{g}c \), and the product of the second row of \( E \) with the \((n + 1)\)-th column of \( M \) is \( \ell' g = \ell' \frac{g}{gc} \tilde{g}c \).

Recall that \( \gcd \{ \tau, \beta \} = 1 \). Note that \( \gcd \{ \frac{\ell m + \ell' (a_1 + b_1)}{h}, \ell' \frac{g}{gc} \} \) divides

\[
-\ell \alpha \frac{m}{h} + \ell' \left( -\alpha \frac{a_1 + b_1}{h} + \frac{g}{g} \right) = -\ell \alpha \frac{m}{h} + \ell' \gcd \{ \frac{a_1 + b_1}{h}, \frac{g}{g} \} = c,
\]

so \( \gcd \{ \frac{\tau \frac{\ell m + \ell' (a_1 + b_1)}{h}, \beta \frac{\ell m + \ell' (a_1 + b_1)}{hc}, \ell' \frac{g}{gc} \} = 1 \), and the second row of \( E \) corresponds to the diagonal entry \( \tilde{g}c \).

Finally, it is obvious that the third row of \( E \) corresponds to the diagonal entry \( h \), and all the other rows correspond to the diagonal entries \( g \). \( \square \)
3. Primitivity

Let \( u, u', v, v' \) be four distinct vertices of \( K_{n,n} \) such that \( u \) and \( u' \) are in the same partite set while \( v \) and \( v' \) are in the other one. Let \( \mu_{(u,v)} \) be a row vector of length \( n^2 \), indexed by the edges of \( K_{n,n} \), such that the entry corresponding to the edge \( \{u, v\} \) is 1 and all other entries are 0. Let \( v_{u,u',v,v'} = \mu_{(u,v)} + \mu_{(v',v)} - \mu_{(u,v')} - \mu_{(u',v')} \). Such a vector is called a 2-pod over the tuple \( (u, u', v, v') \). If \( h \) is a characteristic vector of a nonempty subgraph \( G \) of \( K_{n,n} \), then we say \( G \) or \( h \) is primitive if \( \gcd(vN(G)) = 1 \), where \( v \) is any 2-pod.

**Proposition 4.** The collection of 2-pods \( v_{u,u',v,v'} \) over all tuples \( (u, u', v, v') \) spans over \( \mathbb{Z} \) all the integer vectors in the null space of \( U \).

**Proof.** Let \( w \) be an integer vector in the null space of \( U \), or equivalently \( W \). Let \( \hat{K}_{i,j} \) denote a graph with integer multiplicities on each edge of \( K_{i,j} \) such that the degree at each vertex is 0. Then \( w \) is a \( \hat{K}_{n,n} \), and \( \hat{K}_{2,2} \) is a scalar multiple of some \( v_{u,u',v,v'} \). We will prove that any \( \hat{K}_{i,j} \), \( i, j \geq 2 \), can be decomposed into \( \hat{K}_{2,2} \)'s by induction on \( i + j \).

Let the two partite sets of \( K_{i,j} \) be \( \{u_1, \ldots, u_i\} \) and \( \{v_1, \ldots, v_j\} \). Let \( w(u_i, v_i) \) denote the multiplicity of the edge \( \{u_i, v_i\} \). When \( i = j = 2 \), the decomposition is trivial. Assume that any \( \hat{K}_{i,j} \) can be decomposed into \( \hat{K}_{2,2} \)'s for some \( i, j \geq 2 \). In \( \hat{K}_{i+1,j} \), \( u_1 \) is incident to \( j \) edges with multiplicities \( w(u_1, v_1), \ldots, w(u_1, v_j) \). Then \( w + \sum_{i=2}^j w(u_i, v_i) v_{u_i, u_1, v_i, v_1} \) will still correspond to a \( \hat{K}_{i+1,j} \), but all the edges incident to \( u_1 \) will have multiplicities 0, so this \( \hat{K}_{i+1,j} \) is equivalent to a \( \hat{K}_{i,j} \). By induction hypothesis, \( \hat{K}_{i+1,j} \) can be decomposed into \( \hat{K}_{2,2} \)'s. Similar proof will work for \( \hat{K}_{i,j+1} \). \( \Box \)

**Theorem 5.** If \( h \) is primitive, a set of diagonal factors of \( N \) is

\[
\left( \frac{m g}{4e} \right)^1, (ge)^1, (h)^1, (g)^{2n-4}, (1)^{(n-1)^2},
\]

and a corresponding front can be any unimodular extension \( \tilde{E} \) of \( EU \), where \( E \) is defined in Theorem 3.

We proceed to prove Theorem 5 by first introducing a number of lemmas. The proofs of Lemmas 6 and 7 can be found in [8] and [4] respectively.

**Lemma 6.** Let \( A \) be an \( r \times m \) integer matrix. Suppose \( EA = DA' \) for some unimodular \( E \), diagonal \( D \) and integral \( A' \). Let \( E_i \) be the \( i \)-th row of \( E \) and \( d_i \) be the \( i \)-th diagonal entry of \( D \). If the conditions \( E_i b \equiv 0 \pmod{d_i} \) for \( i = 1, \ldots, r \) are sufficient for the existence of an integer solution \( x \) of \( Ax = b \), then there exists a unimodular matrix \( F \) such that \( EAF = D \).

**Lemma 7.** Given a rational matrix \( A \) and a column vector \( b \), the system \( Ax = b \) has an integer solution \( x \) if and only if for any rational row vector \( y \),

\[
y A \equiv 0 \pmod{1} \text{ implies } y b \equiv 0 \pmod{1}.
\]

**Lemma 8.** If \( h \) is primitive, then any rational row vector \( y \) such that \( y N \equiv 0 \pmod{1} \) implies \( y \equiv z U \pmod{1} \) for some rational vector \( z \).

**Proof.** Let \( v = v_{u,u',v,v'} \) be a 2-pod. Let \( h_{(u,v)}, h_{(u',v')} \) and \( h_{(u,v)(u',v')} \) be the image of \( h \) under the permutations \( (u, v), (u', v') \) and \( (u, v)(u', v') \) respectively. By direct computation, \( (v \cdot h) v^\top = h + h_{(u,v)(u',v')} - h_{(u,v)} - h_{(u',v')} \). As \( h \) is primitive, \( v^\top \) will be in the column module of \( N \) over \( \mathbb{Z} \).
Now, $\mathbf{v}\mathbf{y}^\top = \mathbf{y}\mathbf{v}^\top \equiv 0 \pmod{1}$ since $\mathbf{y}\mathbf{N} \equiv \mathbf{0} \pmod{1}$. Note that $\mathbf{v}$ can run through all 2-pods, so $V\mathbf{y}^\top \equiv \mathbf{0} \pmod{1}$, where $V$ is a matrix whose rows are all the 2-pods $\mathbf{v}$. We claim that there is an integer solution $\mathbf{x}$ to $V\mathbf{x} = V\mathbf{y}^\top$. For any rational row vector $\mathbf{w}$ such that $\mathbf{w}V \equiv \mathbf{0} \pmod{1}$, $\mathbf{w}V$ is an integer vector in the null space of $U$. By Proposition 4, $\mathbf{w}V = \mathbf{w}'V$ for some integer vector $\mathbf{w}'$, so $\mathbf{wV}\mathbf{y}^\top = (\mathbf{w}'V)\mathbf{y}^\top \equiv \mathbf{w}'(V\mathbf{y}^\top) \equiv \mathbf{0} \pmod{1}$. Our claim then follows from Lemma 7.

Let $\mathbf{x}$ be our integer solution to $V\mathbf{x} = V\mathbf{y}^\top$, or $V(\mathbf{y}^\top - \mathbf{x}) = \mathbf{0}$. This implies that $\mathbf{y} - \mathbf{x}^\top$ is in the row space of $U$, so $\mathbf{y} = \mathbf{zU} + \mathbf{x}^\top$ for some rational vector $\mathbf{z}$, i.e. $\mathbf{y} \equiv \mathbf{zU} \pmod{1}$. □

**Lemma 9.** If $\mathbf{h}$ is primitive, $\mathbf{N}\mathbf{x} = \mathbf{b}$ has an integer solution $\mathbf{x}$ if and only if $\mathbf{UN}\mathbf{x}' = \mathbf{Ub}$ has an integer solution $\mathbf{x}'$.

**Proof.** The direction “only if” is trivial. Assume that $\mathbf{x}'$ is an integer solution of $\mathbf{UN}\mathbf{x}' = \mathbf{Ub}$. Let $\mathbf{y}$ be a rational row vector such that $\mathbf{yN} \equiv \mathbf{0} \pmod{1}$. By Lemma 8, $\mathbf{y} \equiv \mathbf{zU} \pmod{1}$ for some rational $\mathbf{z}$. Then $\mathbf{yb} \equiv \mathbf{zUb} = \mathbf{zUN}\mathbf{x}' \equiv \mathbf{yN}\mathbf{x}' \equiv \mathbf{0} \pmod{1}$. By Lemma 7, we are done. □

**Proof of Theorem 5.** Let $d_1 = \frac{mg}{hc}$, $d_2 = \tilde{gc}$, $d_3 = h$ and $d_i = g$ for $i = 4, 5, \ldots, 2n - 1$. As $E$ is a front of $\mathbf{UN}$, there exists a back $\mathbf{F}$ such that $\mathbf{EUNF} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix with diagonal entries $d_i$’s.

Let $\tilde{E}$ be a unimodular extension of $\mathbf{EUN}$ with rows $\tilde{E}_i$, $i = 1, \ldots, n^2$. Suppose $\tilde{E}_i\mathbf{b} \equiv \mathbf{0} \pmod{1}$ for $i = 1, \ldots, 2n - 1$ and $\tilde{E}_i\mathbf{b} \equiv \mathbf{0} \pmod{1}$ for $i = 2n, \ldots, n^2$. Let $\tilde{E}_i\mathbf{b} \equiv \mathbf{0} \pmod{1}$ for $i = 1, \ldots, 2n - 1$, which implies $\mathbf{EUNF} = \mathbf{Dx''}$ for some integer vector $\mathbf{x''}$. Hence, $\mathbf{UN}\mathbf{x'} = \mathbf{Ub}$ has an integer solution $\mathbf{x'} = \mathbf{Fx''}$. By Lemma 9, $\mathbf{N}\mathbf{x} = \mathbf{b}$ has an integer solution $\mathbf{x}$. □

Let $\mathbf{h}$ be primitive. If we multiply a vector $(\ell', -\alpha\frac{m}{hc}, \ell''\alpha\frac{m}{hc} - \ell'\beta\frac{m}{hc}, 0, \ldots, 0)$ to both sides of $\tilde{E}\mathbf{N} = \mathbf{DF}$, where $\tilde{E}$ is the front and $\mathbf{D}$ is the diagonal form in Theorem 5 and $\mathbf{F}$ is a unimodular matrix, then $L.H.S. = 1\mathbf{N} = m\mathbf{I}$, and

$$\text{R.H.S.} = \left(\ell'\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 + (\ell''\alpha\frac{m}{hc} - \ell'\beta\frac{m}{hc})hF_3\right) \equiv \ell''\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 + \beta\frac{m}{c}(\ell''\alpha\frac{m}{hc} - \ell'\beta\frac{m}{hc})hF_3 \equiv \ell''\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 - \beta\frac{m}{hc}F_3,$$

where $F_i$ represents the $i$-th row of $F$. As $L.H.S. = R.H.S.$, we have $\ell''\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 - \beta\frac{m}{hc}F_3 = \mathbf{1}$, which is the unique way to get $\mathbf{1}$ in the row space of $F$ over any $\mathbb{F}_p$ since $\mathbf{F}$ is unimodular. From now on, we use row$_p(A)$ to denote the row space of an integer matrix $A$ over $\mathbb{F}_p$.

**Theorem 10.** If $\mathbf{h}$ is primitive and $p$ is a prime such that $p \mid m$, where $m$ is the number of edges in $G$, then $1$ is in row$_p(\mathbf{N})$ if and only if either of the following holds:

(i) $p \mid h$ and $p \nmid \tilde{g}$,
(ii) $p = 2$, $p \nmid \tilde{g}$ and $p \mid g$,
(iii) $p \neq 2$, $p \nmid \tilde{g}$, $p \mid g$ and $p \mid a_1 + b_1$.

**Proof.** Note that row$_p(\mathbf{N})$ is equal to row$_p(\tilde{E}\mathbf{N})$, which in turn is the same as row$_p(\mathbf{DF})$, so $1$ is in row$_p(\mathbf{N})$ if and only if $\ell''\frac{mg}{hc}F_1 - \alpha\frac{m}{hc}\tilde{g}cF_2 - \beta\frac{m}{hc}F_3$ is in row$_p(\mathbf{DF})$. Again, let $d_1 = \frac{mg}{hc}$, $d_2 = \tilde{gc}$ and $d_3 = h$, which are the first three diagonal entries of $D$. 

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Case 1: \( p \mid h \).

By Lemma 2, \( \beta \) can be chosen in Theorem 3 such that \( \gcd\{\beta, h\} = 1 \), so the coefficient of \( F_3 \) in \( t'\frac{\alpha}{h}F_1 - \frac{\alpha}{h}F_2 - \beta F_3 \) is non-zero in \( \mathbb{F}_p \). However, \( F_3 \) is not in row \( p(DF) \) since \( d_3 = 0 \) in \( \mathbb{F}_p \). Therefore, \( 1 \) is not in row \( p(\mathbb{N}) \).

Case 2: \( p \nmid h \) and \( p \mid g \).

In this case, \( F_3 \) is in row \( p(DF) \). Note that \( \frac{g}{h} = \frac{a}{h(\frac{a}{h})} \) which divides \( \frac{a}{h} \), so the coefficients of \( F_1 \) and \( F_2 \) in \( t'\frac{\alpha}{h}F_1 - \frac{\alpha}{h}F_2 - \beta F_{n+1} \) are 0’s in \( \mathbb{F}_p \). Therefore, \( 1 \) is in row \( p(\mathbb{N}) \).

Case 3: \( p \nmid g \).

Again, \( F_3 \) is in row \( p(DF) \). As \( \tilde{g} = \gcd\{a_1 - b_1, g\} \), we have \( p \nmid a_1 - b_1 \). If \( p = 2 \), then \( p \nmid a_1 + b_1 \), so \( p \nmid c \), and hence the coefficient of \( F_1 \) in \( t'\frac{\alpha}{h}F_1 - \frac{\alpha}{h}F_2 - \beta F_{n+1} \) is 0 in \( \mathbb{F}_p \). Also, \( F_2 \) is in row \( p(DF) \) since \( d_2 = \tilde{g}c \) which is non-zero in \( \mathbb{F}_p \). Therefore, \( 1 \) is in row \( p(\mathbb{N}) \).

If \( p \neq 2 \), if \( p \nmid a_1 + b_1 \), then by the same argument, \( 1 \) is in row \( p(\mathbb{N}) \). If \( p \mid a_1 + b_1 \), then \( p \mid c \), so \( F_2 \) is not in row \( p(DF) \). However, \( \gcd\{\alpha, \frac{a_1+b_1}{h}, \frac{g}{h}\} = 1 \) implies \( p \nmid \alpha \), so the coefficient of \( F_2 \) in \( \tilde{t}'\frac{\alpha}{h}F_1 - \frac{\alpha}{h}F_2 - \beta F_{n+1} \) is non-zero in \( \mathbb{F}_p \). Therefore, \( 1 \) is not in row \( p(\mathbb{N}) \).

Case 4: \( p \mid g \).

As \( p \nmid c \), \( \ell \gcd\{\frac{a_1+b_1}{h}, \frac{g}{h}\} - \ell \alpha = c \) implies \( p \nmid \ell \), so the coefficient of \( F_1 \) in \( \ell'\frac{\alpha}{h}F_1 - \frac{\alpha}{h}F_2 - \beta F_3 \) is non-zero in \( \mathbb{F}_p \). However, \( F_1 \) is not in row \( p(DF) \) since \( d_1 = 0 \) in \( \mathbb{F}_p \). Therefore, \( 1 \) is not in row \( p(\mathbb{N}) \).

\[ \square \]

4. NON-PRIMITIVE CASE

Proposition 11. Let \( G \) be a nonempty subgraph of the complete bipartite graph \( K_{n,n} \) with \( 2n \) vertices. Then \( G \) is non-primitive if and only if \( G \) is \( K_{n,n}, K_{s,n} \cup \{n - s \text{ isolated vertices}\} \) or \( K_{s,t} \cup K_{n-s,n-t} \) for some \( 1 \leq s, t \leq n - 1 \).

Proof. Let \( \{u_1, \ldots, u_n\} \) and \( \{v_1, \ldots, v_n\} \) be two partite sets of \( G \). Let \( u \leftrightarrow v \) denote \( \{u, v\} \) is an edge in \( G \) while \( u \leftrightarrow v \) denote \( \{u, v\} \) is not.

If \( G \) is \( K_{n,n} \), then it is obviously non-primitive. If \( G \) is non-primitive but not \( K_{n,n} \), then without loss of generality, \( u_1, u_2, v_1, v_2 \) satisfy one of the following two cases:

(a) \( u_1 \leftrightarrow v_1 \) and \( u_1 \leftrightarrow v_2 \), while \( u_2 \leftrightarrow v_1 \) and \( u_2 \leftrightarrow v_2 \);
(b) \( u_1 \leftrightarrow v_1 \) and \( u_2 \leftrightarrow v_2 \), while \( u_1 \leftrightarrow v_2 \) and \( u_2 \leftrightarrow v_1 \).

If (a) occurs, then for any \( u \neq u_1, u_2 \), either \( u \leftrightarrow v_i \) for both \( i \in \{1, 2\} \) or \( u \leftrightarrow v_i \) for both \( i \in \{1, 2\} \), so \( \Gamma(v_i) = \Gamma(v_2) \), where \( \Gamma(x) \) denotes all the neighbors of vertex \( x \). Now, for any \( v \neq v_1, v_2 \), either \( v \leftrightarrow u \) for all \( u \in \Gamma(v_1) \) and \( v \leftrightarrow u' \) for all \( u' \notin \Gamma(v_1) \), or \( v \leftrightarrow u \) for all \( u \in \Gamma(v_1) \) and \( v \leftrightarrow u' \) for all \( u' \notin \Gamma(v_1) \). Hence, \( G \) is either \( K_{s,n} \cup \{n - s \text{ isolated vertices}\} \) or \( K_{s,t} \cup K_{n-s,n-t} \) for some \( 1 \leq s, t \leq n \).

If (b) occurs, then for any \( u \neq u_1, u_2 \), exactly one of \( u \leftrightarrow v_1 \) and \( u \leftrightarrow v_2 \) occurs. Note that \( \Gamma(v_1) \) and \( \Gamma(v_2) \) form a partition of \( \{u_1, \ldots, u_n\} \). Now, for any \( v \neq v_1, v_2 \), either \( v \leftrightarrow u \) for all \( u \in \Gamma(v_1) \) and \( v \leftrightarrow u' \) for all \( u' \in \Gamma(v_2) \), or \( v \leftrightarrow u \) for all \( u \in \Gamma(v_1) \) and \( v \leftrightarrow u' \) for all \( u' \in \Gamma(v_2) \). Hence, \( G \) is \( K_{s,t} \cup K_{n-s,n-t} \) for some \( 1 \leq s, t \leq n \).

\[ \square \]

Theorem 12. If \( G \) is non-primitive, then diagonal forms of \( G \) can be given by the following.
Proof. (a) is trivial.
(b) and (c) The diagonal forms in these cases are obtained by explicit integral row and column operations. Please refer to [10] for the details.

\[ \text{Theorem 13.} \text{ If } h \text{ is non-primitive, } 1 \text{ is in row}_{p}(N) \text{ if and only if either of the following holds, where } p \mid m \text{ in each case:} \]

(i) \( G \) is \( K_{n,n} \),
(ii) \( G \) is \( K_{s,n} \sqcup \{n-s \text{ isolated vertices} \} \) with \( p \nmid s \),
(iii) \( G \) is \( K_{s,t} \sqcup K_{n-s,n-t} \) with \( p \nmid \epsilon \) and \( p \mid 2\lambda \delta \), where \( \epsilon = \gcd\{n,s,t\} \), \( \lambda = \gcd\{n-2s,t-s\} \), \( \delta = \gcd\{\frac{n-t+s}{\epsilon}, 2\} \).

Proof. (i) Every row of \( N \) is 1, so 1 is in row\(_{p}(N)\).
(ii) By keeping track of the row operations in the proof of part (b) in Theorem 12, a front \( E \) of \( N \) is

\[
\begin{array}{cccc}
P & Q & \cdots & Q \\
R & S & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
R & S & & \\
\end{array}
\]

where

\[
P = \begin{bmatrix}
1 & 1_{n-2} & -(n-2) \\
0_{n-2} & I_{(n-2)\times(n-2)} & -1_{n-2}^T \\
0 & 0_{n-2} & 1 \\
\end{bmatrix}, \quad Q = \begin{bmatrix}
O_{n\times(n-1)} & 1 \\
0_{n-2}^T & \\
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
-I_{(n-1)\times(n-1)} & 1_{n-1}^T \\
0_{n-1} & 0 \\
\end{bmatrix}, \quad S = \begin{bmatrix}
I_{(n-1)\times(n-1)} & -1_{n-1}^T \\
0_{n-1} & 1 \\
\end{bmatrix}
\]

and there are \( n \) horizontal sections. The first row of \( E \) corresponds to the diagonal entry \( s \), the \( n \)-th row corresponds to the diagonal entry \( \epsilon \), the second to the \((n-1)\)-th row and the \((in)\)-th row, \( 2 \leq i \leq n \), correspond to the diagonal entry 1, and the rest corresponds to 0.
Let $EN = DF$. Note that the first row of $EN$ is $s\mathbf{1}$, so $F_1 = \mathbf{1}$, where $F_1$ is the first row of $F$. Therefore, $\mathbf{1}$ is in row $p(N)$ if and only if $p \nmid s$ since the first entry of $D$ is $s$.

(iii) A front $E$ of $N$ in part (c) of Theorem 12 is

\[
\begin{array}{|c|c|c|c|}
\hline
P' & P' & \ldots & P' & Q' \\
\hline
R' & & & S' & \\
\hline
\vdots & \ddots & & \vdots & \\
\hline
R' & & & S' & \\
\hline
T' & T' & \ldots & T' & U' \\
\hline
\end{array}
\]

where

\[
P' = \begin{bmatrix}
1 & \mathbf{1}_{n-2} & \omega_1 \\
0_{n-2}^\top & I_{(n-2)\times(n-2)} & -\mathbf{1}_{n-2} \\
0 & \mathbf{0}_{n-2} & 1+\mu
\end{bmatrix},
\]

\[
Q' = \begin{bmatrix}
\omega_2 & \omega_2\mathbf{1}_{n-2} & \omega_3 \\
0 & (1-(n-2s))I_{(n-2)\times(n-2)} & ((n-2s)-1)\mathbf{1}_{n-2}^\top \\
\mu & \mu\mathbf{1}_{n-2} & \omega_4
\end{bmatrix},
\]

\[
R' = \begin{bmatrix}
1 & \mathbf{1}_{n-2} & 1-(n-2s) \\
0_{n-2}^\top & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^\top \\
0 & \mathbf{0}_{n-2} & 1
\end{bmatrix},
\]

\[
S' = \begin{bmatrix}
-1 & -\mathbf{1}_{n-2} & (n-2s)-1 \\
0_{n-2}^\top & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^\top \\
0 & \mathbf{0}_{n-2} & 1
\end{bmatrix},
\]

\[
T' = \begin{bmatrix}
1 & \beta(1+\mu) \\
O_{n\times(n-1)} & \mathbf{0}_{n-1}^\top
\end{bmatrix},
\]

\[
U' = \begin{bmatrix}
1 - \frac{\nu}{\epsilon} \mathbf{1}_{n-2} & (1 - \frac{\nu}{\epsilon} \mu)\mathbf{1}_{n-2} & \omega_5 \\
0_{n-2}^\top & I_{(n-2)\times(n-2)} & \mathbf{1}_{n-2}^\top \\
0 & \mathbf{0}_{n-2} & 1
\end{bmatrix},
\]

and $m = (n-s)(n-t) + st$, $\mu = \frac{2\nu + \phi + \xi - 1}{2}$, $\nu = n - t + s$, $\omega_1 = 1 - \frac{m}{\epsilon} \left[ 1 + (1 - \frac{\nu}{\epsilon})(1 + \mu) \right]$, $\omega_2 = 1 - \frac{m}{\epsilon} \left[ 1 + (1 - \frac{\nu}{\epsilon}) \mu \right]$, $\omega_3 = 1 - \frac{m}{\epsilon} (\omega_4 + \omega_5)$, $\omega_4 = 1 - (n-2s) + (2 - (n-2s))\mu$, $\omega_5 = 2 - (n-2s) - \frac{\nu}{\epsilon} \omega_4$. The first row of $E$ corresponds to the diagonal entry $\frac{2m\lambda}{\epsilon^2}$, the $n$-th row corresponds to the diagonal entry $\epsilon$, the $(n(n-1)+1)$-th row corresponds to the diagonal
entry $\delta \lambda$, the second to the $(n-1)$-th row and the $(1+n)$-th row, $1 \leq i \leq n-2$, correspond to the diagonal entry $2\lambda$, the $(i+jn)$-th row, $2 \leq i \leq n-1$, $1 \leq j \leq n-2$, correspond to the diagonal entry 2, while the $(in)$-th row, $2 \leq i \leq n-1$, and the $((n-1)n+2)$-th to the last row correspond to the diagonal entry 1.

This matrix works as a front since the first, $n$-th and $(n(n-1)+1)$-th rows come from the row operations in the proof of part (c) of Theorem 12. As for the other rows, we can multiply to $N$ directly to check.

Now, if we multiply $\begin{pmatrix} 1,0,\ldots,0,\frac{m}{\tau},0,\ldots,0,\frac{m}{\tau},0,\ldots,0 \end{pmatrix}$ to both sides of $EN = DF$, we will get $m1 = \frac{2m\lambda}{\epsilon} F_1 + \epsilon F_n + \frac{m}{\tau} \delta \lambda F_{n(n-1)+1}$, where $F_i$ denotes the $i$-th row in $F$. By dividing $m$ from both sides of the equation, we have $1 = \frac{2\lambda}{\epsilon} F_1 + F_n + \frac{\delta \lambda}{\epsilon} F_{n(n-1)+1}$, which is the unique way to obtain 1 in the row space of $F$ over any field $\mathbb{Z}_p$ since $F$ is unimodular.

Case 1: $p \mid \epsilon$.

The coefficient of $F_n$ in $\frac{2\lambda}{\epsilon} F_1 + F_n + \frac{\delta \lambda}{\epsilon} F_{n(n-1)+1}$ is non-zero in $\mathbb{F}_p$. However, $F_n$ is not row$_p(DF)$ since the $n$-th entry of $D$ is $\epsilon$ which is 0 in $\mathbb{F}_p$. Therefore, 1 is not in row$_p(N)$.

Case 2: $p \nmid \epsilon$ and $p \nmid \frac{2\lambda}{\epsilon}$.

The coefficient of $F_1$ in $\frac{2\lambda}{\epsilon} F_1 + F_n + \frac{\delta \lambda}{\epsilon} F_{n(n-1)+1}$ is non-zero in $\mathbb{F}_p$. However, $F_1$ is not in row$_p(DF)$ since the first entry of $D$ is $\frac{2m\lambda}{\epsilon}$ which is 0 in $\mathbb{F}_p$. Therefore, 1 is not in row$_p(N)$.

Case 3: $p \mid \epsilon$ and $p \nmid \frac{2\lambda}{\epsilon}$.

The coefficient of $F_1$ in $\frac{2\lambda}{\epsilon} F_1 + F_n + \frac{\delta \lambda}{\epsilon} F_{n(n-1)+1}$ is 0 in $\mathbb{F}_p$. As $p \nmid \epsilon$, $F_n$ is in row$_p(DF)$. If $p \mid \delta \lambda$, then the coefficient of $F_{n(n-1)+1}$ is also 0 in $\mathbb{F}_p$; if $p \nmid \delta \lambda$, then $F_{n(n-1)+1}$ is in row$_p(DF)$ since the $(n(n-1)+1)$-th entry of $D$ is $\delta \lambda$. Therefore, 1 is in row$_p(N)$.

5. ZERO-SUM (MOD 2) BIPARTITE RAMSEY NUMBERS

Let $G$ be a simple nonempty bipartite graph with $m$ edges. A $p$-coloring on the edges of $G$ is a function $c: E(G) \rightarrow \mathbb{Z}_p$. If $\sum_{e \in E(G)} c(e) = 0$ over $\mathbb{Z}_p$, then we say that $G$ is a zero-sum (mod $p$) graph with respect to $c$. If $p \mid m$, then the zero-sum bipartite Ramsey number $B_p(G)$ is the smallest integer $n$ such that for every $p$-coloring of $K_{n,n}$, there exists a zero-sum (mod $p$) copy of $G$ in $K_{n,n}$.

Zero-sum Ramsey problems are first studied by Bialostocki and Dierker [2] and Alon and Caro [1], and the zero-sum (mod 2) bipartite Ramsey numbers are fully characterized by Caro and Yuster [3] in the following theorem. We are going to provide our own proof here.

**Theorem 14 ([3]).** Let $G$ be a simple nonempty bipartite graph with even number of edges. Let $n$ be the minimum number such that the vertices of $G$ can be divided into two partite sets, each of size not exceeding $n$. Then isolated vertices are added to $G$ if necessary to make each partite set of $G$ have size $n$. Let $B_2(G)$ denote the zero-sum bipartite Ramsey number of $G$ modulo 2. Then $B_2(G) = n + 1$ if and only if one of the following holds:

(i) $G$ is primitive with all degrees odd,

(ii) $G$ is primitive such that any representation of $G$, where each partite set has size $n$, have all degrees in one partite set odd and all degrees in the other partite set even,

(iii) $G = K_{n,n}$,

(iv) $G = K_{s,n} \sqcup \{n-s \text{ isolated vertices}\}$ with $s$ odd,

(v) $G = K_{s,t} \sqcup K_{n-s,n-t}$ with $n$ even and at least one of $s$ and $t$ is odd.

Otherwise, $B_2(G) = n$. #
Proof. Let \( p = 2 \). By Theorems 10 and 13, the vector of all 1’s \( \mathbf{1} \) is in row\(_2\)(\( N \)) if and only if one of these five cases hold. We will check the only unobvious case, which is Theorem 13(iii) implying Theorem 14(v). If \( n \) is odd, then \( \lambda \) is also odd. As \( p \nmid \epsilon \) and \( p \mid \frac{2s}{\delta} \), \( s \) and \( t \) are of the same parity, which contradicts that \( p \) divides \( m = st + (n - s)(n - t) \). If \( n \) is even, then \( p \) always divides \( m \). Since \( p \nmid \epsilon \), at least one of \( s \) and \( t \) is odd. If \( s \) and \( t \) are of opposite parity, then \( \delta \) is odd and \( p \mid \frac{2s}{\delta} \). If both \( s \) and \( t \) are odd, then \( \lambda \) is even and again \( p \mid \frac{2s}{\delta} \).

\( B_2(G) > n \) if and only if there exists a 2-coloring on the edges of \( K_{n,n} \) such that all isomorphic copies of \( G \) in \( K_{n,n} \) have color sum equal to 1 (mod 2). In other words, \( \mathbf{1} \) is in row\(_2\)(\( N \)), which happens if and only if one of these five cases hold. Note that when two more isolated vertices are added to \( G \) so that \( G \) is embedded in \( K_{n+1,n+1} \), none of these five cases hold, so we always have \( B_2(G) \leq n + 1 \). Combining these two directions, this theorem is proved. \( \square \)

6. Signed Bipartite Graph Design

Let \( G \) be a nonempty proper subgraph of the complete bipartite graph \( K_{n,n} \) with \( 2n \) vertices, some of which are possibly isolated. Let \( G \) be the collection of subgraphs \( G' \) of \( K_{n,n} \) which are isomorphic to \( G \). We say that there exists a \((n, G, \lambda)\)-signed bipartite graph design if there exists \( z : G \to \mathbb{Z} \) such that for each edge \( e \in E(K_{n,n}) \),

\[
\sum_{G' \in G : E(G') = e} z(G') = \lambda.
\]

If \( \lambda = 1 \) and \( z : G \to \{0, 1\} \), then this problem is related to graph decompositions, studied by Wilson [7] and many others. Ushio [5] gives the necessary and sufficient conditions for a complete bipartite graph to be decomposed into complete bipartite graphs. Here, the necessary and sufficient conditions for the existence of a \((n, G, \lambda)\)-signed bipartite graph design are given.

Theorem 15. Let \( G \) be a nonempty proper subgraph of the complete bipartite graph \( K_{n,n} \) with \( 2n \) vertices. If \( G \) has only one connected component of size greater than 1, then there exists a \((n, G, \lambda)\)-signed bipartite graph design if and only if all the following three conditions hold:

(i) \( h \mid \lambda n \),
(ii) \( \gcd \mid \lambda n(\ell n + 2\ell' + \ell'') \),
(iii) \( \frac{mg}{\mathbb{m}} \mid \lambda n(\alpha + \beta + \frac{g}{\mathbb{m}}) \) + \( \alpha \frac{2n}{h} + \beta \frac{mg}{\mathbb{m}} \),
where \( \alpha, \beta,\ell,\ell' \) and \( \ell'' \) are defined in Theorem 3.

Proof. Note that there exists a \((n, G, \lambda)\)-signed bipartite graph design if and only if there exists an integer solution \( z \) to \( N(G)z = \lambda \mathbf{1}^T \). If \( G \) is primitive, then by Theorem 5, \( \mathbb{E}N\mathbb{F} = D \) for some unimodular matrix \( F \), where \( D \) contains the set of diagonal factors given in Theorem 5. So \( \mathbb{E}^{-1}DF^{-1}z = \lambda \mathbf{1}^T \), or \( Dz' = \lambda \mathbb{E}^1 \mathbf{1}^T \) for some integer solution \( z' \), which exists if and only if \( d_i \mid \lambda E_i \mathbf{1}^T \) for \( i = 1, 2, \ldots, 2n - 1 \), where \( E_i \) is the \( i \)-th row of \( E \) given in Theorem 3.

By definition, \( \mathbf{1}^T = (n^2, n, \ldots, n)^T \), so \( \lambda E_i \mathbf{1}^T = 0 \) which is divisible by \( d_i \) for \( i \geq 4 \). When \( i = 3, 2 \) and \( 1 \), they correspond to the conditions (i), (ii) and (iii) respectively.

If \( G \) is non-primitive, then \( G = K_{s,n} \cup \{n - s \text{ isolated vertices}\} \). The conditions (i) to (iii) combine to be \( \frac{h}{\mathbb{m}} \mid \lambda \), which is equivalent to \( s \mid \lambda n \) since \( h = \gcd\{n, s\} \), and \( s \mid \lambda n \) is the condition obtained from the diagonal form in Theorem 13. \( \square \)
Corollary 16. If \( G = K_{s,t}, 1 \leq s \leq t \leq n \), then there exists a \((n, G, \lambda)\)-signed bipartite graph design if and only if \( st \mid \lambda n^2 \).

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References