1. In class, we discussed the “Gambler’s Ruin” problem, in which a gambler who starts with $a$ dollars bets a dollar each time step on a bet which pays off with probability $p$, against a house with $b$ dollars. That is, if $X_n$ is the number of dollars the gambler has at time $n$, then $X_{n+1} = X_n$ if $X_n = 0$ or $X_n = a + b$ (i.e., either the gambler or the house has no money), and otherwise $X_{n+1} = X_n + B_n$, where $B_n$ is an independent random variable which is 1 with probability $p$ and $-1$ with probability $1 - p$. We then showed that the process terminates with probability 1, and determined the expected time it takes for the process to terminate.

In this problem, we consider a variant, in which the house has an infinite supply of money (so $X_{n+1} = X_n$ only if $X_n = 0$). Show that (a) if $p > 1/2$, the process terminates with probability strictly between 0 and 1, (b) if $p < 1/2$, the process terminates with probability 1, and the time it takes has bounded expectation, and (c) determine what happens when $p = 1/2$.

2. (3.3.12) Using the identity $\sin(t) = 2\sin(t/2) \cos(t/2)$ repeatedly suggests the formula

$$\frac{\sin(t)}{t} = \prod_{1 \leq m} \cos(t/2^m).$$

Prove the latter identity by interpreting both sides as characteristic functions.

3. (3.3.20) If $Y_n$ are random variables with characteristic functions $\phi_n$, then $Y_n \Rightarrow 0$ iff there exists $\delta > 0$ such that $\phi_n \rightarrow 1$ pointwise on the interval $(-\delta, \delta)$.

4. (3.4.2) Let $X_1, X_2, \ldots$ be i.i.d. with $E(X) = 0$, $0 < \text{Var}(X) < \infty$, and let $Y_n = n^{-1/2} \sum_{i=1}^n X_n$. Use the central limit theorem and Kolmogorov’s zero-one law to conclude that $\limsup_{n \to \infty} Y_n = \infty$ a.s.

5. (3.6.7) Let $T_n$ be the time of the $n$th arrival in a rate $\lambda$ Poisson process, let $U_1, \ldots, U_n$ be i.i.d. uniform on $(0, 1)$, and let $V_1 \leq V_2 \leq \cdots \leq V_n$ be the corresponding sorted sequence. Show that the vectors $\vec{V}$ and $(T_1/T_{n+1}, \ldots, T_n/T_{n+1})$ have the same distribution.

6. (3.8.3) Show that if $\mu$ is infinitely divisible, its characteristic function is never 0. (The book suggests an argument using (3.3.20) above)

7. (5.1.6) Give an example of a random variable and a pair of $\sigma$-algebras such that $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) \neq E(E(X|\mathcal{F}_2)|\mathcal{F}_1)$ with probability 1.

8. (5.2.11) Let $X_n$ and $Y_n$ be sequences of positive integrable random variables, and $\mathcal{F}_n$ a sequence of $\sigma$-algebras such that $X_n$ and $Y_n$ are both measurable w.r.t. $\mathcal{F}_n$. Suppose that $\sum_{n=1}^\infty Y_n < \infty$ a.s., and that

$$E(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n)X_n,$$

so that $X_n$ is only approximately a supermartingale. Show that $X_n$ still converges a.s. to a finite limit. (Hint: Reduce to the Martingale Convergence Theorem.)