Ma/CS 6b
Class 10: Kuratowski's Theorem

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Plane Graphs

• A plane graph is a drawing of a graph in the plane such that the edges are non-crossing curves.
Planar Graphs

- The drawing on the left is not a plane graph. However, on the right we have a different drawing of the same graph, which is a plane graph.
- An abstract graph that can be drawn as a plane graph is called a planar graph.

Non-Planar Graphs

- Recall. We proved that $K_5$ and $K_{3,3}$ are not planar.
  - Thus, every graph that has $K_5$ or $K_{3,3}$ as a subgraph is also not planar.
  - Are there graphs that do not contain $K_5$ and $K_{3,3}$ as a subgraph, and are not planar?
    - Yes, and we can use $K_5$ and $K_{3,3}$ to generate them.
More Non-Planar Graphs

- **Subdividing edges** of $K_5$ or $K_{3,3}$ cannot make them planar.
  - If we have a plane drawing after the subdivision, the same drawing works for the original graph.

Reminder: Graph Subdivision

- $G$ is a **subdivision** of $H$.
- We refer to the added vertices as **subdivision vertices**.
  - These vertices are of degree 2.
Reminder: Topological Minors

- A graph $H$ is a topological minor of a graph $G$ if $G$ contains a subdivision of $H$ as a subgraph.

Kuratowski's Theorem

- **Theorem.** A graph is planar if and only if it does not have $K_5$ and $K_{3,3}$ as topological minors.
  - We know that if a graph contains $K_5$ or $K_{3,3}$ as a topological minor, then it is not planar.
  - It remains to prove that every non-planar graph contains such a topological minor.

Kazimierz Kuratowski
Minimal Non-planar Graph

- A **minimal non-planar graph** is a non-planar graph $G$ such that any proper subgraph of $G$ is planar.
- What minimal non-planar graphs can you think of?
  - $K_5$ and $K_{3,3}$.

Kuratowski Subgraphs

- Given a graph $G$, a **Kuratowski subgraph** of $G$ is a subgraph that is a subdivision of $K_5$ or of $K_{3,3}$.
Proof Strategy

• To prove Kuratowski’s theorem, we need to prove that every non-planar graph contains a Kuratowski subgraph.
  ◦ It suffices to prove this only for minimal non-planar graphs.
  ◦ We will show that every minimal non-planar graph with no Kuratowski subgraph must be 3-connected.
  ◦ We then show that every 3-connected graph with no Kuratowski subgraph is planar.

Contradiction!

Choosing the Unbounded Face

• Lemma. Let $G$ be a planar graph, and let $F$ be a set of edges of $G$ that form the boundary of a face in an embedding of $G$. Then there exists a non-crossing drawing of $G$ where $F$ is the boundary of the unbounded face.
Proof

- We draw the graph on a sphere, and then project it from a point on the face $f$.
  - In the projection on the plane, $f$ will be the unbounded face.

Bad Math Joke #1

- **Q:** What do you call a young eigensheep?
- **A:** A lamb, duh!
Connectedness of Minimal Non-planar Graphs

- **Claim.** A minimal non-planar graph must be **1-connected**.
  - Assume, for contradiction, that there exists a minimal non-planar graph $G$ that is not connected.
  - Let $C$ be one connected component of $G$.
  - By the minimality of $G$, both $C$ and $G - C$ are planar.
  - But then we can draw $C$ and then draw $G - C$ inside one of its faces. **Contradiction!**

2-Connectedness

- **Claim.** A minimal non-planar graph must be **2-connected**.
  - Assume, for contradiction, that there exists a minimal non-planar graph $G = (V, E)$ that is not 2-connected.
  - There exists a vertex $v$ whose removal disconnects $G$.
  - Let $C$ be a component of $G - v$.
  - By the minimality of $G$, the induced subgraph on $C \cup \{v\}$ and $(V \setminus C) \cup \{v\}$ are both planar.
  - We can embed both graphs with $v$ on the unbounded face, and merge both copies of $v$. 
Preparing for 3-Connectedness

- **Claim.** Let $G \in (V, E)$ be a non-planar graph and let $x, y \in V$, such that $G \setminus \{x, y\}$ is disconnected. Then there is a component $C$ of $G - \{x, y\}$ such that the induced subgraph on $C \cup \{x, y\}$ with the edge $(x, y)$ is non-planar.
Proof

- $C_1, \ldots, C_k$ – the components of $G - \{x, y\}$.
- $G'_i$ – the induced subgraph on $C_i \cup \{x, y\}$, plus the edge $(x, y)$.
- Assume, for contradiction, that $G'_1, \ldots, G'_k$ are all planar.
  - $H_1$ – a plane drawing of $G'_1$.
  - $H_i$ (for $2 \leq i \leq k$) – drawing $G'_i$ (without crossings) in a face of $H_{i-1}$ with $(x, y)$ on its boundary, and merging the two copies of $x, y$.
  - Each $H_i$ is planar, including $H_k = G$.

Contradiction!

3-Connectedness

- **Lemma.** Let $G = (V, E)$ be a graph with fewest edges among all non-planar graphs without a Kuratowski subgraph. Then $G$ is 3-connected.

- **Proof.**
  - $G$ is obviously a minimal non-planar graphs.
  - By a previous lemma, $G$ is 2-connected.
  - It remains to prove that there are no two vertices $x, y \in V$ such that $G - \{x, y\}$ is disconnected.
Proof

- Assume, for contradiction, that there exist $x, y \in V$ such that $G - \{x, y\}$ is disconnected.
  - $C_1, \ldots, C_k$ – the components of $G - \{x, y\}$.
  - By the previous lemma, there exists $C_i$ such that the induced subgraph on $C_i \cup \{x, y\}$ plus the edge $(x, y)$ is non-planar. Denote it as $H$.
  - By the minimality of $G$, we have that $H$ have a Kuratowski subgraph $K$.
  - Since $G$ does not contain $K$, it must be that $(x, y) \in K$ and $(x, y) \not\in E$.

Proof (cont.)

- Let $C'$ be another connected component of $G - \{x, y\}$.
- In $G$ there is a path $P$ between $x$ and $y$ that uses only vertices of $C'$.
- Combining $P$ with the other edges of $K$ yields a Kuratowski subgraph of $G$. **Contradiction!**
Bad Math Joke #2

• **Q:** What do you get when you cross a mountain goat and a mountain climber?
• **A:** Nothing—you can’t cross two scalars.

Recap

• We proved that a smallest non-planar graph without a Kuratowski subgraph is 3-connected.
  ◦ To complete the proof of [Kuratowski's Theorem](#), we prove that every 3-connected graph without a Kuratowski subgraph is planar.
Contraction Cannot generate Kuratowski Subgraphs

• **Lemma.** Let $G = (V, E)$ be a graph with no Kuratowski subgraph. Then contracting any edge $e \in E$ does not result in a Kuratowski subgraph.
  
  ◦ $G_e$ – the graph that is obtained by contracting $e = (x, y)$ in $G$.
  
  ◦ Assume, for contradiction, that $G_e$ contains a Kuratowski subgraph $H$.

Proof

• $v_e$ – the vertex in $G_e$ that is obtained by contracting $e = (x, y)$.

• If $v_e$ is not in $H$, then $H$ is also a subgraph of $G$. **Contradiction!**

• $v_e$ cannot have degree zero or one in $H$.

• If $v_e$ has degree two in $H$, we can obtain $H$ in $G$ by replacing $v_e$ with $x$ (or $y$). **Contradiction!**
Proof (cont.)

- Consider the case where $v_e$ has degree $d_v \geq 3$ in $H$, and after expanding $e$ back $x$ (or $y$) is of degree $\geq d_v$ in $G$.
  - Then $H$ is also in $G$ with $x$ replacing $v_e$ and $y$ being a subdivision vertex. **Contradiction!**

![Diagram]

Proof (cont.)

- A single case remains: $H$ is a subdivision of $K_5$ and after expanding $e$ back both $x$ and $y$ are of degree 3.
  - In this case $G$ contains $K_{3,3}$. **Contradiction!**
  - In the figure, we have $y, a, b$ on one side and $x, c, d$ on the other.
Bad Math Joke #3

- **Q:** What do you get if you cross an elephant and a banana?
- **A:** $|\text{elephant}| \cdot |\text{banana}| \cdot \sin \theta$.

Contractions and 3-Connectivity

- **Lemma.** Let $G = (V, E)$ be a 3-connected graph with $|V| \geq 5$. Then there exists an edge $e \in E$ whose contraction results in a 3-connected graph.
Proof

Assume, for contradiction, that there exists a 3-connected \( G = (V, E) \) with \( |V| \geq 5 \), such that contracting any \( e \in E \) yields a graph \( G_e \) that is not 3-connected.

- For any \( e \in E \), let \( v_e \) denote the vertex of \( G_e \) to which \( e \) is contracted.
- Since \( G_e \) is not 3-connected, there exists \( z_e \in V \) such that \( G_e - \{v_e, z_e\} \) is disconnected.

Proof (cont.)

- Every \( e = (x, y) \in E \) has \( z_e \in V \) such that:
  - \( G_e - \{v_e, z_e\} \) is disconnected.
  - \( G - \{x, y, z_e\} \) is disconnected
- We choose an edge \( e = (x, y) \) so that the size of the largest component \( C \) of \( G - \{x, y, z_e\} \) is maximized.
  - \( C' \) – another component of \( G - \{x, y, z_e\} \).
  - There must be an edge \( f \) between \( z_e \) and a vertex \( u \in C' \).
Proof (cont.)

- Let $C'$ be another component. There is an edge $f$ between $z_e$ and a vertex $u \in C'$.
  - By definition, $G - \{z_e, u, z_f\}$ is disconnected.
  - The induced subgraph of $C \cup \{x, y\}$ is connected. Also, deleting $z_f$ from this subgraph cannot disconnect it, since this would imply that $G - \{z_e, z_f\}$ is disconnected (but $G$ is 3-connected!).
  - So $G - \{z_e, u, z_f\}$ is disconnected and contains a component larger than $C$.

Contradiction!

The End

- A physicist and a mathematician are sitting in a faculty lounge. Suddenly, the coffee machine catches on fire. The physicist grabs a bucket and leaps toward the sink, fills the bucket with water, and puts out the fire.

- Another day, and the same two sit in the same lounge. Again the coffee machine catches on fire. This time, the mathematician stands up, gets a bucket, and hands the bucket to the physicist, thus reducing the problem to a previously solved one.